CM20019 – Complementary Course Notes

Sheet 1

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1 Substitutions and Unifiers

1.1 Definition. A *substitution* θ is a mapping from the set of variables into the set of terms which is equal to the identity mapping almost everywhere except for a finite set of variables, i.e., $\{X \mid \theta(X) \neq X\}$ is finite. θ is represented as the finite set of pairs $\{(X_1, t_1), \ldots (X_n, t_n)\}$, where $X_1, \ldots, X_n$ are different variables and for all $1 \leq i \leq n$ we find $X_i \neq t_i$. An alternative representation is the following (adopted in the course notes): $[X_1 := t_1, \ldots, X_n := t_n]$. Substitutions are denoted by $\theta, \eta, \ldots$ The identity mapping, i.e., the *empty substitution*, is denoted by ϵ. A pair (X_i, t_i) (or, equivalently, $X_i := t_i$) occurring in a substitution is called a *binding*. If a substitution θ is a one-to-one and onto mapping from its domain to itself, then θ is called a *renaming*.

Let $\theta = [X_1 := t_1, \ldots, X_n := t_n]$ be a substitution and V a set of variables;

- $\texttt{Dom}(\theta)$ denotes the set $\{X_1, \ldots, X_n\}$.
- $\texttt{Range}(\theta)$ the set $\{t_1, \ldots, t_n\}$.
- $\texttt{VRange}(\theta)$ denotes the set of variables occurring in $\texttt{Range}(\theta)$.
- $\texttt{Var}(\theta) = \texttt{Dom}(\theta) \cup \texttt{VRange}(\theta)$.
- $\theta\mid_V = \{X := t \in \theta \mid X \in V\}$ is the *restriction* of θ to V.

1.2 Example. Consider the substitution $\theta = [X := f(Y, Z), Y := g(h(W), a)]$. Then,
$\texttt{Dom}(\theta) = \{X, Y\}$,
$\texttt{Range}(\theta) = \{f(Y, Z), g(h(W), a)\}$,
$\texttt{VRange}(\theta) = \{W, Y, Z\}$,
$\texttt{Var}(\theta) = \{W, X, Y, Z\}$,
$\theta\mid_{X,W} = [X := f(Y, Z)]$.
θ is not a renaming substitution, but $\sigma = [X := Y, Y := Z, Z := X]$ is.

1.3 Definition. The result of *applying a substitution θ to a term s*, denoted by $s\theta$ is the term obtained by simultaneously replacing each occurrence of a variable from $\texttt{Dom}(\theta)$ in s by the corresponding term in $\texttt{Range}(\theta)$. $s\theta$ is called an *instance* of s under θ. θ is said to be a *grounding substitution* for a term s if $s\theta$ is ground. In this case, $s\theta$ is also called a ground instance of s. If θ is a renaming, then $s\theta$ is called a *variant* of s. Finally, substitutions are extended as morphisms on literals and sets of literals in the obvious way.

1.4 Example. Consider the substitutions θ and σ given above and the term $s = g(X, g(Y, Z))$. Then,
$s\theta = g(f(Y, Z), g(g(h(W), a), Z))$.
$s\sigma = g(Y, g(Z, X))$.
Neither $s\theta$ nor $s\sigma$ is a ground instance of s.
$g(X, g(Y, Z))$ and $g(Y, g(Z, X))$ are variants.

1.5 Definition. The *composition* $\theta \circ \eta$ of the substitutions θ and η is the substitution whose bindings are in the following set:

$$\{X := t\eta \mid X := t \in \theta, X \neq t\eta\} \cup \{Y := t \mid Y := t \in \eta, Y \notin \mathtt{Dom}(\theta)\}.$$

(Notation: sometimes the symbol $\circ$ is omitted)

The composition of substitutions is associative, has ϵ as the left and right identity and the equation $(E\theta)\eta = E(\theta\eta)$ holds for any expression E.

1.6 Example. Consider again the substitutions θ and σ given above, then
$\theta\sigma = [X := f(Z, X), Y := g(h(W), a), Z := X]$,
$\sigma\theta = [X := g(h(W), a), Y := Z, Z := f(Y, Z)]$.

1.7 Definition. The substitution θ is said to be more general then the substitution τ if for some substitution η we find $\tau = \theta\eta$.

This definition is not as simple as it looks like and the interested reader may consult e.g. [Apt97][1] for further reading.

1.8 Example. $\theta = [X := f(Y, Z)]$ is more general than $\eta = [X := f(a, b), Y := a, Z := b]$, because with $\tau = [Y := a, Z := b]$ we find that $\theta\tau = \eta$. However, θ is not more general than $\sigma = [X := f(a, b)]$. On the other hand, $\theta\tau|_X = [X := f(a, b)] = \sigma$.

1.9 Definition. A *unification problem* consists of two terms s and t and is the question whether there exists a substitution θ such that $s\theta = t\theta$. If such a θ exists, then θ is said to be a *unifier* for s and t. θ is called *most general unifier*, in short mgu, for s and t if it is a unifier for s and t which is more general than all unifiers for s and t. An mgu θ of s and t is called *strong* if for all unifiers η of s and t we find $\eta = \theta\eta$.

The unification problem is decidable. If two terms s and t are unifiable, then there exists a strong mgu for s and t which is unique modulo variable renaming, and several algorithms are known for computing *the* strong mgu. Finally, unification problems and their solutions can be extended to pairs of atoms in the obvious way.

(Please refer to the course notes for examples.)

[1] K.R. Apt. *From Logic to Logic Programming.* Prentice Hall, London, 1997.