

Unit Code: CM20019 - S1 (Year 2006/07)
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Worksheet 2:
Predicate Logic: Syntax, Semantics, Normal Forms
(29/10/2006)

Exercise 1

Express in first order predicate logic the sentence “At night all cats seem to be black.”

Exercise 2

A prime number is a natural number n such that there are no two natural numbers different than 1 whose product is n . Define an interpretation I and a formula F such that

$$(\forall X)(p(X) \leftrightarrow F),$$

where $p^I = \{n \mid n \text{ is a prime natural number}\}$. Justify your answer.

Exercise 3

Consider the following formulae in a language with predicate symbols $p/1$ and $r/2$:

$$\begin{aligned} G_1 &= (\exists X)(\forall Y)(\exists Z)((p(X) \rightarrow r(X, Y)) \wedge p(Y) \wedge \neg r(Y, Z)) \\ G_2 &= (\exists X)(\exists Z)((r(Z, X) \rightarrow r(X, Z)) \rightarrow (\forall Y)r(X, Y)) \\ G_3 &= (\forall Y)((\exists Z)(\forall U)r(U, Z) \wedge (\forall X)(r(X, Y) \rightarrow \neg r(X, Y))) \\ G_4 &= (\exists X)(\forall Y)((p(Y) \rightarrow r(Y, X)) \wedge ((\forall U)(p(U) \rightarrow r(U, Y)) \rightarrow r(X, Y))) \\ G_5 &= (\forall X)(\forall Y)((p(X) \wedge r(X, Y)) \rightarrow \\ &\quad ((p(Y) \wedge \neg r(Y, X)) \rightarrow (\exists Z)(\neg r(Z, X) \wedge \neg r(Y, Z)))) \\ G_6 &= (\forall Z)(\forall U)(\exists X)(\forall Y)((p(U) \wedge r(X, Y)) \rightarrow (p(Y) \wedge r(Z, X))) \end{aligned}$$

For each of these formulae, determine whether or not it is satisfied in each of the following structures:

- the base set is \mathbf{N} , the interpretation of r is the order relation \leq , and the interpretation of p is the set of even integers.
- the base set is $\wp(\mathbf{N})$ (the set of subsets of \mathbf{N}), the interpretation of r is the inclusion relation \subseteq , and the interpretation of p is the collection of finite subsets of \mathbf{N} .
- the base set is \mathbf{R} , the interpretation of r is the set of pairs $(a, b) \in \mathbf{R}^2$ such that $b = a^2$, and the interpretation of p is the subset of rational numbers.

Exercise 4

Prove that:

1. $(\exists X)(F \rightarrow G) \equiv ((\forall X)F \rightarrow (\exists X)G)$, for all formulae F and G .
2. $(\exists X)(G \rightarrow F) \equiv (G \rightarrow (\exists X)F)$, if X is not free in G .
3. $(\forall X)(G \rightarrow F) \equiv (G \rightarrow (\forall X)F)$, if X is not free in G .
4. $(\exists X)(F \rightarrow G) \equiv ((\forall X)F \rightarrow G)$, if X is not free in G .

5. $(\forall X)(F \rightarrow G) \equiv ((\exists X)F \rightarrow G)$, if X is not free in G .

Exercise 5

Prove that $(\exists X)(F \wedge G) \rightarrow ((\exists X)F \wedge (\exists X)G)$ is a valid formula.

Exercise 6

Given the formula $F = (\forall X, Y)(\exists Z)(\forall U)(p(X, Z) \wedge p(Y, Z) \rightarrow q(X, Z, U)) \wedge (\forall Z)r(X, Z)$, produce

- an equivalent prenex normal form;
- a formula G where no universal quantifiers are present and which is valid iff F is valid;
- a formula G where no existential quantifiers are present and which is unsatisfiable iff F is unsatisfiable.