Unit Code: CM20019 - S1 (Year 2006/07)
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Worksheet 2:
Predicate Logic: Syntax, Semantics, Normal Forms
(29/10/2006)

Exercise 1
Express in first order predicate logic the sentence "At night all cats seem to be black."

Exercise 2
A prime number is a natural number n such that there are no two natural numbers different than 1 whose product is n. Define an interpretation I and a formula F such that

$$(\forall X)(p(X) \leftrightarrow F),$$

where $p^I = \{\, n \mid n \text{ is a prime natural number}\}$. Justify your answer.

Exercise 3
Consider the following formulae in a language with predicate symbols $p/1$ and $r/2$:

$$\begin{array}{ll} G_1 = & (\exists X)(\forall Y)(\exists Z)((p(X) \rightarrow r(X,Y)) \wedge p(Y) \wedge \neg r(Y,Z)) \\ G_2 = & (\exists X)(\exists Z)((r(Z,X) \rightarrow r(X,Z)) \rightarrow (\forall Y) r(X,Y)) \\ G_3 = & (\forall Y)((\exists Z)(\forall U) r(U,Z) \wedge (\forall X)(r(X,Y) \rightarrow \neg r(X,Y))) \\ G_4 = & (\exists X)(\forall Y)((p(Y) \rightarrow r(Y,X)) \wedge ((\forall U)(p(U) \rightarrow r(U,Y)) \rightarrow r(X,Y))) \\ G_5 = & (\forall X)(\forall Y)((p(X) \wedge r(X,Y)) \rightarrow \\ & \quad ((p(Y) \wedge \neg r(Y,X)) \rightarrow (\exists Z)(\neg r(Z,X) \wedge \neg r(Y,Z)))) \\ G_6 = & (\forall Z)(\forall U)(\exists X)(\forall Y)((p(U) \wedge r(X,Y)) \rightarrow (p(Y) \wedge r(Z,X))) \end{array}$$

For each of these formulae, determine whether or not it is satisfied in each of the following structures:

- the base set is $\mathbf{N}$, the interpretation of r is the order relation $\leq$, and the interpretation of p is the set of even integers.
- the base set is $\wp(\mathbf{N})$ (the set of subsets of $\mathbf{N}$), the interpretation of r is the inclusion relation $\subseteq$, and the interpretation of p is the collection of finite subsets of $\mathbf{N}$.
- the base set is $\mathbf{R}$, the interpretation of r is the set of pairs $(a,b) \in \mathbf{R}^2$ such that $b = a^2$, and the interpretation of p is the subset of rational numbers.

Exercise 4
Prove that:

1. $(\exists X)(F \rightarrow G) \equiv ((\forall X)F \rightarrow (\exists X)G)$, for all formulae F and G.
2. $(\exists X)(G \rightarrow F) \equiv (G \rightarrow (\exists X)F)$, if X is not free in G.
3. $(\forall X)(G \rightarrow F) \equiv (G \rightarrow (\forall X)F)$, if X is not free in G.
4. $(\exists X)(F \rightarrow G) \equiv ((\forall X)F \rightarrow G)$, if X is not free in G.

5. $(\forall X)(F \rightarrow G) \equiv ((\exists X)F \rightarrow G)$, if X is not free in G.

Exercise 5
Prove that $(\exists X)(F \wedge G) \rightarrow ((\exists X)F \wedge (\exists X)G)$ is a valid formula.

Exercise 6
Given the formula $F = (\forall X, Y)(\exists Z)(\forall U)(p(X, Z) \wedge p(Y, Z) \rightarrow q(X, Z, U)) \wedge (\forall Z)r(X, Z)$, produce

- an equivalent prenex normal form;
- a formula G where no universal quantifiers are present and which is valid iff F is valid;
- a formula G where no existential quantifiers are present and which is unsatisfiable iff F is unsatisfiable.