CM 20019

COURSEWORK 2 - due 16 NOV 2006

SAMPLE SOLUTIONS

EXERCISE 1

Rewrite the formulae according to the given interpretation, and evaluate them in the domain $\mathbb{N}$. Justify if the statements are true in $\mathbb{N}$, or, if they are false, provide a counterexample:

1. $S_1^I = (\forall x)\ \neg(x > x)$ true, because '>' is not reflexive

   $S_2^I = (\forall x)(\exists y)\ x > y$ false, for $x = 0$

   $S_3^I = (\forall x)(\forall y)(\forall z)(x > y \wedge y > z \rightarrow x > z)$ true, because '>' is transitive

   So the three statements are not simultaneously true in the given interpretation

2. $S_1^I = (\forall x)\ \neg\ (x \leq x)$ false, because '$\leq$' is reflexive

   $S_2^I = (\forall x)(\exists y)\ x \leq y$ true, because $\mathbb{N}$ is a well-founded set

   $S_3^I = (\forall x)(\forall y)(\forall z)(x \leq y \wedge y \leq z \rightarrow x \leq z)$ true, because '$\leq$' is reflexive

   So the three statements are not simultaneously true in the given interpretation.

EXERCISE 2

1. The domain $\mathbb{N}$ is split into even and not-even (odd) numbers, and this splitting is induced by the given p^I. We need then to consider two cases:

   if x ~~:= m~~, m even, $(\forall x)\ (p(x) \rightarrow p(f(x)))$ is such that $p^I(x) = T$ and $p^I(f^I(x)) = p^I(x+4) = T$.

   Hence under any assignment $\sigma = [x := m]$ where m is an even number

   $$\left[(\forall x)(p(x) \rightarrow p(f(x)))\right]^{I,\sigma} = T \rightarrow T = T$$

if $X := m$, m not-even, $(\forall x)(p(x) \to p(f(x)))$ is such

that $p^I(x) = F$ and $p^I(f^I(x)) = p^I(x+4) = F$

Hence, under any assignment $\sigma = [x := m]$

where m is a not-even number

$$[(\forall x)(p(x) \to p(f(x)))]^{I,\sigma} = F \to F = T$$

There are no possible other assignments for X, because the partitioning of $\mathbb{N}$ into even and not-even is complete.

It follows that I is a model for the given S.

2. One possibility (there are other too) is to define
$f^{I'}(x) = a$, $\forall x \in D$. In this case

$$\left.\begin{array}{lll} \text{for } x=a & : & p(a) \to p(a) \\ \text{for } x=b & : & p(b) \to p(a) \\ \text{for } X=c & : & p(c) \to p(a) \end{array}\right\} \text{are all evaluated to } T$$

EXERCISE 3:

$S = (A \leftrightarrow (B \vee C)) \leftrightarrow (B \vee \neg C)$

Write down the truth table:

A	B	C	$A \leftrightarrow (B \vee C)$	$(B \vee \neg C)$	S
0	0	0	1	1	1
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	0	1	0
1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	1	1	1

- DNF : Each row where S is 1 identifies an element of the DNF. The inputs A, B, C for such rows are to be taken in conjunction, preserving their sign.

$$(\neg A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C) \vee (A \wedge B \wedge C).$$

- CNF: Each row where S is 0 identifies one element of the CNF. The inputs A, B, C for such rows are to be taken in disjunction, and if the input is 0, the atom will occur positively in the disjunction, whereas if the input is 1, the atom will occur negated.

$$(A \vee \neg B \vee C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee C) \wedge (\neg A \vee B \vee \neg C)$$

- Clausal form:
  (for example, in set notation)

  $\{\{A, \neg B, C\} \ \{A, \neg B, \neg C\} \ \{\neg A, B, C\} \ \{\neg A, B, \neg C\}\}$

  Alternatively, as implications:

  $(B \to A \vee C) \quad \wedge$
  $(B \wedge C \to A) \quad \wedge$
  $(A \to B \vee C) \quad \wedge$
  $(A \wedge C \to B) \quad .$

EXERCISE 4

Recall the definition of prenex normal form:
A formula F is prenex iff there exist a $k \in \mathbb{N}$ and variables $w_1, \dots, w_k$ such that $F = (Q_1 w_1) \cdots (Q_k w_k) G$, where Q_i are quantifiers $1 \le i \le k$ and G is a formula quantifier-free.

The proof is by induction on the structure of F.

- F is atomic. Then let $G \equiv \forall F$ be the universal closure of F. We know that $F \equiv G$ (equivalence law).

- $F = \neg F_1$. By inductive hypothesis F_1 is equivalent to a formula in pnf
  $(Q_1 v_1) \dots (Q_k v_k) G_1$, where G_1 is

quantifier-free.
Consider the formula

$$G_1' = (\overline{Q_1} v_1) \cdots (\overline{Q_k} v_k) \neg G_1$$

obtained by considering for $1 \le i \le k$ $(\overline{Q_i})$ as the dual quantifier of Q_i.
Then $F \equiv G_1'$ and G_1' is in pnf.

- $F = (F_1 \vee F_2)$

By inductive hp. there exists formulae in pnf equivalent to F_1 and F_2 respectively. Let

$G_1 = (Q_1 v_1) \cdots (Q_k v_k) G_1'$ be a pnf

and $F_1 \equiv G_1$ and

$G_2 = (Q_1' z_1) \cdots (Q_h' z_h) G_2'$ be a pnf

and $F_2 \equiv G_2$

Consider $k+h$ pairwise distinct variables $x_1, \ldots, x_k, y_1 \ldots, y_h$ which do not occur neither in G_1' nor in G_2'.
Consider now the variants G_1^* and G_2^* of G_1' and G_2', respectively:

$$G_1^* = G_1' [v_1 := x_1, \ldots v_k := x_k]$$

$$G_2^* = G_2' [z_1 := y_1 \ldots z_h := y_h]$$

Let G be

$$G = (Q_1 x_1) \cdots (Q_k x_k)(Q_1' y_1) \cdots (Q_h' y_h) (G_1^* \vee G_2^*)$$

By the equivalence laws, we see that $G \equiv F$ and G is in pnf.

- $F = (F_1 \wedge F_2)$
- $F = (F_1 \to F_2)$

similar to the previous case

- $F = (\exists v) F_1$. By i.h. $F_1 \equiv G_1$ and G_1 is in pnf.
  Let $G_1 = (Q_1 w_1) \dots (Q_k w_k) G_1'$
  where G_1' is quantifier free.

  Two cases are possible:

  1. $v \notin \{w_1 \dots w_k\}$.
     Take $G = (\exists v) G_1$;
     Then $G \equiv F$ and G is pnf.
  2. $v \in \{w_1 \dots w_k\}$
     Hence v is not free in $G_1 = (Q_1 w_1) \dots (Q_k w_k) G_1'$.
     Consider an α-renaming of G_1'
     $(Q_1 w_1') \dots (Q_k w_k') G_1''$
     so that variables are all distinct and $v \notin \{w_1' \dots w_k'\}$. Apply as in case 1.

- $F = (\forall v) F_1$. Similar to the previous case.

EXERCISE 5:

Rename all variables apart:

$(\forall Z)(\exists Y)\left(p(U, g(Y), Z) \vee \neg(\forall X) q(X)\right) \wedge \neg (\forall V)(\exists W) \neg r(f(W,V),V))$

Propagate negation:

$(\forall Z)(\exists Y)\left(p(U, g(Y), Z) \vee (\exists X) \neg q(X)\right) \vee (\exists V)(\forall W)\, r(f(W,V),V))$

Move quantifiers to the front, apply equivalence laws:

$(\forall Z)(\exists Y)(\exists X)(\exists V)(\forall W)\left[\left(p(U, g(Y), Z) \vee \neg q(X)\right) \wedge r(f(W,V),V)\right]$

Eliminate $\exists Y$: introduce Skolem function $f_1/1$:

$(\forall z)(\exists x)(\exists v)(\forall w)\left[(p(u, g(f_1(z)), z) \vee \neg q(x)) \wedge r(f(w,v),v)\right]$

Eliminate $\exists x$: introduce skolem function $f_2/1$:

$(\forall z)(\exists v)(\forall w)\left[(p(u, g(f_1(z)), z) \vee \neg q(f_2(z))) \wedge r(f(w,v),v)\right]$

Eliminate $\exists v$: introduce Skolem function $f_3/1$:

$(\forall z)(\forall w)\left[p(u, g(f_1(z)), z) \vee \neg q(f_2(z))) \wedge r(f(w, f_3(z)), f_3(z))\right]$

EXERCISE 6:

By hp: $F \equiv \neg G$ (1)

By completeness thm $\models F \leftrightarrow \neg G$.

By hp: F is satisfiable i.e. there exists an interpretation I such that $\models_I F$. (2)

Hence $\models_I \neg G$

By hp: F is not valid i.e. there exists an interpretation J such that $\not\models_J F$. Therefore $\models_J \neg F$. From (1) and using de Morgan laws, it follows $\models_J G$. Whence G is satisfiable.

We now show that G can't be valid. We reason by contradiction.

Assume that G is valid. This means, by definition of validity, that for all interpretation k, $\models_k G$. Then, by (1) and de Morgan laws it follows that for all k $\models_k \neg F$, i.e. $\neg F$ is valid. By theorem relationship validity and unsatisfiability (see Deduction thm) it follows that F is unsatisfiable. This contradicts the given hp. (2).

EXERCISE 7 :

1. $F\left[((\forall x)\,p(x) \lor (\forall x)\,q(x)) \to (\forall x)\,(p(x) \lor q(x))\right]$

2. $T\left[(\forall x)\,p(x) \lor (\forall x)\,q(x)\right]$ (1)

3. $F\left[(\forall x)\,(p(x) \lor q(x))\right]$ (1)

4. $F\left[p(c) \lor q(c)\right]$, c new (3)

5. $F\left[p(c)\right]$ (4)

6. $F\left[q(c)\right]$ (4)

7. $T\left[(\forall x)\,p(x)\right]$ (2)

9 $T\left[p(c)\right]$ (7)

X contradiction

5, 9

8. $T\left[(\forall x)\,q(x)\right]$ (2)

10. $T\left[q(c)\right]$ (10)

X contradiction

6, 10

EXERCISE 8

$$F = (\forall x)(\forall z)\Big[\Big(\underbrace{\big(p(x) \to (\exists y)\,q(y)\big)}_{= F_1} \land \underbrace{p(x)}_{= F_2} \land \underbrace{\big((\exists y)\,q(y) \to r(z)\big)}_{= F_3}\Big) \to r(z)\Big]$$

Rename variables apart in F_3 and simplify main '$\to$':

$$F \equiv (\forall x)(\forall z)\left[\neg\left(F_1 \land F_2 \land ((\exists u)\,q(u) \to r(z))\right) \lor r(z)\right]$$

Simplify implications, propagate negation

$$\equiv (\forall x)(\forall z)\Big[\big(p(x) \land (\forall y)\neg q(y)\big) \lor \neg p(x) \lor \big((\exists u)\,q(u) \land \neg r(z)\big) \lor r(z)\Big]$$

Move quantifiers to the front (equivalence laws)

$$(\forall X)(\forall Z)(\forall Y)(\exists U)\left[\left(p(X) \wedge \neg q(Y)\right) \vee \neg p(X) \vee \left(q(U) \wedge \neg r(Z)\right) \vee r(Z)\right]$$

F valid iff $\neg F$ unsatisfiable.

$$\neg F \equiv (\exists X)(\exists Z)(\exists Y)(\forall U)\left[\left(\neg p(X) \vee q(Y)\right) \wedge \left(p(X)\right) \wedge \left(\neg q(U) \vee r(Z)\right) \wedge \neg r(Z)\right]$$

Skolemise, preserving unsatisfiability. Introduce constants a_X, a_Z, a_Y; the skolem form of $\neg F$ is:

$$(\forall U)\left[\left(\neg p(a_X) \vee q(a_Y)\right) \wedge p(a_X) \wedge \left(\neg q(U) \vee r(a_Z)\right) \wedge \neg r(a_Z)\right]$$

Clausal form:

$\{\neg p(a_X), q(a_Y)\}$ $\{p(a_X)\}$ $\{\neg q(U), r(a_Z)\}$ $\{\neg r(a_Z)\}$

$\{q(a_Y)\}$

$\{r(a_Z)\}$ $\sigma = [U := a_Y]$

$\Box$

mgu : $\sigma = [U := a_Y]$