DESIGNING A PROOF SYSTEM
AROUND A NORMALISATION PROCEDURE

[REPORT ON A 3-YEAR PROJECT ON
EFFICIENT AND NATURAL PROOF SYSTEMS]

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INVITED TALK AT ALCOP 2016 (WIEN)
WITH A FEW MORE SLIDES FOR PCC 2016 (MÜNCHEN)

TALK AVAILABLE AT
http://cs.bath.ac.uk/ag/t/DPSANP.pdf
Gentzen Proof Theory

Compression? Manipulation? Locality? Modularity?...
... Names! Scope! Quantifiers! Substitutions! Trees!

In Gentzen:

\[ \langle \text{Formula} \rangle \rightarrow \langle \text{Tautology} \rangle \]

The formula tree gives shape to the proof tree.

Good

Not good enough
Deep inference | — Better trees

A better proof tree looks like this:

Vertical composition: inference
Horizontal composition: any (!) connective

It's more general than Gentzen trees and yields:

- Exponential speed-ups
- Analyticity for modal and exotic logics
- Canonical forms
- Computational interpretations for concurrency
... and much more
Deep Inference 2 - Trivial Cut Elimination

Assignent $A_\sigma = \exists \lambda \beta \forall \alpha \lambda \alpha \lambda \beta \lambda f \lambda \ldots$

Applying $\sigma$:

\[
\begin{array}{c}
\text{Cut} \\
\frac{B \land \overline{B}}{f}
\end{array}
\]

\[
\mathbf{B_\sigma \land \overline{B_\sigma}} = \frac{f \land t}{f}
\]

The cut disappears

Proof $\varphi$ with cuts:

Conclusion of $\varphi$ $\Rightarrow$ C

Apply all $\sigma_1, \ldots, \sigma_n$

Canonical proof with no cuts:

\[
\begin{array}{c}
A_{\sigma_1} \\
\varphi \circ \sigma_1 \\
\text{No cuts} \\
\lor \ldots \lor \\
C
\end{array}
\]

\[
\begin{array}{c}
A_{\sigma_2^n} \\
\varphi \circ \sigma_2^n \\
\text{No cuts} \\
C
\end{array}
\]

Several contractions

C

No cuts - Canonical Form !!!
## Results of this Project

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↑

Algebraic freedom
(Process Algebras)

↑

Separation
Simplification
Design Choices: Separation
Simplification

- Just a linear core for splitting

- Just outside of Gentzen possibilities: seq (self-dual non-commutative connective)

- Simplest possible scoping mechanism for contraction: self-duality
Design choices: Separation Simplification 2

- Just a linear core for splitting

- Just outside of Gentzen possibilities: SEQ (self-dual non-commutative connective)

- Simplest possible scoping mechanism for contraction: self-duality

\[ BV + \star = KV \]
Design choices: separation simplification

- Just a linear core for splitting
  
- Just outside of Gentzen possibilities:
  \[ \text{seq} (\text{self-dual non-associative connective}) \]

- Simplest possible scoping mechanism for contraction: self-duality

Competitor: NEL = BV + ! + ?
  \[ \rightarrow \text{complex!} \]

\[ \text{BV} + * = \boxed{\text{KV}} \]
Normalisation in NEL

\[
\begin{align*}
\text{p} & \downarrow [!] a \otimes a \\
\text{b} & \downarrow ![?! a \otimes !a] \\
\text{b} & \uparrow ![?! a \otimes !a] \\
\text{p} & \uparrow (!?! a \otimes !a) \\
\text{p} & \uparrow [?](!a \otimes !a) \\
\text{INDUCTION MEASURE:} & \text{THE ONION!}
\end{align*}
\]
Atomic Flows — Locality Yields Topology

\[
\frac{t}{a \lor \tilde{a}}
\]

\[
\frac{m}{[a \lor t] \land [t \lor \tilde{a}]}
\]

\[
\frac{s}{[a \lor t] \land \tilde{a}}
\]

\[
\frac{s}{a \land \tilde{a} \lor t}
\]

\[
\frac{f}{a \land \tilde{a} \lor t}
\]

Atomic Flows From Proofs

- Only structural information is retained
- Logical information is lost
- Size is polynomially related to proof size

→ Enough to control normalisation !!!
Complexity Generation

Mechanism:

\[ \rightarrow \]

\[ \rightarrow \text{Exponential Blow-up} \]

- Not necessarily easy to control
- Exponential complexity also generated by substitution and a simpler use of contraction

The Sausage!
KV Mechanism I: Contraction

$\text{SEQ: SELF-DUAL NON-COMMUTATIVE}$

$A \triangleleft \star A$  

$A \triangleleft \star A$  

$\times A$  

$A \triangleleft \star A$

$\sim \text{KLEENE STAR}$

$(I_1 \triangleleft I_2) \triangleleft \star P \rightarrow (I_1 \triangleleft I_2) \triangleleft P \triangleleft \star P$

$\rightarrow (I_1 \triangleleft P) \triangleleft (I_2 \triangleleft \star P)$

$\rightarrow \emptyset_1 \triangleleft (I_2 \triangleleft \star P)$

$\rightarrow \ldots$

$k\left\{ A \triangleleft \star A \right\} \triangleleft \star k\left\{ A \triangleleft \star A \right\}$

$\star k\{ \star A \}$

$\rightarrow \text{EXPONENTIAL COMPRESSION}$
KV Mechanism 2: Cut

Identity / Cut

\[
\frac{0}{A \otimes \overline{A}} \quad \frac{A \otimes \overline{A}}{0}
\]

Collapsed Identity

\[A \otimes 0 = A \otimes 0 = 0 \otimes A = A \otimes 0 = A\]

Cut Elimination by Splitting: i.e. 'the context mimicks the formula'

The result can be generalised to linear logics
Decomposition test

\[
\frac{A \circ \star A}{\star A} \otimes \star \overline{A} \quad \rightarrow \quad \frac{(A \circ \star A) \otimes \star \overline{A}}{\overline{A} \circ \star \overline{A}}
\]

\[
\rightarrow \quad \text{Good!}
\]
DECOMPOSITION TEST 2

\[
\frac{A \triangleleft \star A}{\star A} \quad \rightarrow \quad A \triangleleft \star A
\]

\[
\frac{A \triangleleft \star A}{A \triangleleft \star A}
\]

\[\rightarrow \quad \text{NO SAUSAGES — NO ONIONS!}
\]

\[\text{VERY GOOD!}\]
DECOMPOSITION

NATURAL NOTION OF COMPUTATION, E.G. CONFLUENT
KV MECHANISM 3: SUBSTITUTION

\[
\text{\small \begin{align*}
\frac{a \otimes \bar{a}}{0} / \{ a \leftarrow \frac{A \triangle * A}{* A} \} &= \\
\left\{ \frac{A \triangle * A}{* A} \otimes \frac{\bar{A}}{* \bar{A}} \right\} &\approx \left\{ (A \triangle * A) \otimes \frac{\bar{A}}{* \bar{A}} \right\}
\end{align*}}\]

'FORMALISM B' EQUIVALENCE

EXPONENTIAL COMPRESSMION MECHANISM NEEDS TO BE INDEPENDENT OF

* DECOMPOSITION ... OK
* SPLITTING (LINEAR CUT ELIMINATION) ... →
INDEPENDENCE OF CUT ELIMINATION AND SUBSTITUTION

\[
\text{CUT} \quad \frac{}{A \quad \{a \leq B\}} \quad C
\]

\[
\text{C.E.} \quad \frac{}{A \quad \{a \leq B\}} \quad C
\]

DOESN'T WORK

NOT C.E.!
INDEPENDENCE OF CUT ELIMINATION
AND SUBSTITUTION 2

\[
\text{cut} \quad / \quad \{ 2 \leftarrow \Box \}
\]

\[
\begin{align*}
\text{A} & \quad \text{C.E.} \\
\text{B} & \quad \text{C.E.}
\end{align*}
\]

WORKS!

We need to make identity atomic:

\[
\begin{align*}
\text{A} & \quad \Box \quad \Box \quad \Box \\
\text{A} & \quad \Box \quad \Box \quad \Box
\end{align*}
\]

\[
\begin{align*}
\ast (A \& B) & \quad \ast A \uparrow \ast B \\
\ast (A \& B) & \quad \ast A \uparrow \ast B
\end{align*}
\]

\[
\ast (A \& B) \rightarrow \text{SELF DUAL } \kappa (Ripke)
\]

\[
\ast A \uparrow \ast B \rightarrow \ast (A \& B) \rightarrow \ldots
\]
Proof System KV

\[
\frac{\neg \neg \top}{\top} \quad \frac{(A \circ C) \circ (B \circ D)}{(A \circ B) \circ (C \circ D)} \quad \frac{\neg A \circ \neg B}{\neg (A \circ B)} \quad \frac{\top}{A \circ \neg A}
\]

\[
\frac{\top}{\neg \neg \top} \quad \frac{(A \circ B) \circ C}{A \circ (B \circ C)} \quad \frac{(A \circ B) \circ (C \circ D)}{(A \circ C) \circ (B \circ D)} \quad \frac{\neg (A \circ B)}{\neg A \circ B} \quad \frac{A \circ \neg A}{\neg A}
\]

- Equations: ASS., CONN., UN., \( \neg \circ = \circ \)

Draft paper on request
Moral Results of the Project

• There are several, independent normalisation mechanisms.

• Cut elimination in Gentzen is the conflation of two of them: splitting and decomposition.

• Gentzen’s proof theory suffers from a massive artefact: a weak tree structure.

(KV is nice, but only a technical example.)

• In deep inference, computation drives the design of proof systems, not the other way round!
References

BV [1]; NEL [4, 5]; sausages [2, 3].

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