

# Subatomic Proof Systems

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The proof theoretic methodology of deep inference [5] yields the widest range of analytic proof systems. In particular, several logics for which there are no analytic proof systems in Gentzen, or for which there only are cumbersome ones, admit elegant and regular analytic proof systems in deep inference. This regularity on the inference rules, coupled with the surprising observation that if we allow rules to see ‘inside atoms’ then even disparate rules such as contraction, cut, identity and logical rules like conjunction-introduction can be made to fit a unique rule scheme, are the basis of what we call *subatomic logic* [1].

By employing this new methodology, we are able to present logical systems in such a way that every rule is an instance of the rule scheme

$$\frac{(A \alpha B) \nu (C \beta D)}{(A \nu C) \alpha (B \gamma D)},$$

where  $\alpha, \nu, \beta, \gamma$  are relations, and  $A, B, C, D$  are formulae. We call such systems subatomic. There exist subatomic systems for a wide variety of logics— classical logic, linear logic and BV [6] for example— many of which are substructural in the sense that contractions are controlled. In this talk, we will present some of the main results obtained by exploiting the unprecedented regularity of subatomic systems. We will show that in many logics we can separate proofs into a purely linear phase where contractions and weakenings are absent, followed by a phase made-up of contractions and weakenings only. Further, we will present a general cut-elimination technique that can be applied to a wide range of substructural logics without contractions and weakenings [2]. Last, we will present some work in progress in which we exploit the ability to access the ‘inside’ of atoms to turn unit equations of the form  $A \nu f = A$  into linear rules, giving us a novel way to understand equations and adding to the regularity of subatomic systems.

The main idea behind subatomic logic is to consider atoms as logical relations, and to build formulae by freely composing constants by connectives and atoms. For example,  $A \equiv (\mathbf{f} \mathbf{a} \mathbf{t}) \nu \mathbf{t}$  is a subatomic formula for classical logic. We will then translate subatomic formulae into ‘regular’ formulae through an interpretation map that will interpret  $(\mathbf{f} \mathbf{a} \mathbf{t})$  as a positive occurrence of the atom  $a$ , and  $(\mathbf{t} \mathbf{a} \mathbf{f})$  as a negative occurrence of the same atom, denoted by  $\bar{a}$ . Intuitively, we can view subatomic formulae as a superposition of truth values. For example,  $(\mathbf{f} \mathbf{a} \mathbf{t})$  is the superposition of the two possible assignments for the atom  $a$ , and  $(\mathbf{t} \mathbf{a} \mathbf{f})$  is the superposition of the possible assignments for  $\bar{a}$ : if we read the value on the left of the atom we assign  $f$  to  $a$  and  $t$  to  $\bar{a}$ , and viceversa if we read the one on the right. By developing this methodology,

we are able to present subatomic proof systems where every rule has the same shape, and to use the interpretation map to translate proofs in these subatomic systems into proofs in the ‘usual’ systems.

In many systems, derivations can be arranged into consecutive subderivations made up of only certain rules [4], [8], [9]. We call this property *decomposition*. In subatomic systems, since there is only a single inference rule shape to consider, we are able to generally study the interactions between rules. We can thus provide generalised reduction rules to manipulate proofs through local transformations to obtain their decomposition into a linear phase without contractions and weakenings, followed by a phase made up only of these rules.

The reach of the decomposition procedure as a normalisation technique is made apparent when combined with a particular cut-elimination method. In the sequent calculus, formulae have a root connective that allows us to determine which rules are applied immediately above the cut. In deep inference, rules can be applied anywhere deep in a formula and as such anything can happen above a cut. As a consequence, normalisation techniques in deep inference focus on understanding the behaviour of the context around the cut. The *splitting* method in particular can be used to show cut-elimination for many different logics [3], [6], [7], [9]. By exploiting the regularity of subatomic systems, we generalise the splitting theorem to provide sufficient conditions for a system to enjoy cut-elimination. This result can be applied to a wide range of substructural logics without contractions and weakenings [2].

In this way, splitting deals with the interactions between cuts and linear non-contraction rules, whereas decomposition deals with the interactions between cuts and contractions. These two phenomena, tangled in traditional Gentzen-style cut-elimination procedures through the use of a mix rule conflating cuts and contractions, turn out to be quite different complexity-wise. Splitting is a procedure of polynomial-time complexity where we need to look at a whole proof in order to eliminate the cuts, whereas decomposition has an exponential cost and can be achieved through local rewritings. By untangling these interactions and separating cut-elimination into these two procedures we can therefore gain a better control on the complexity, as well as a better understanding of the reasons behind the prevalence of cut-elimination in such a width of proof systems.

As well as exploiting the perspective of subatomic logic for the regularity of the inference rules, we can exploit the interpretation map as a mechanism that allows us to preserve

information in the subatomic proofs that will not be observable once interpreted into an ‘ordinary’ proof. We aim to use this ability to turn equations of the form  $A \vee f = A$  where a unit disappears into ‘linear’ derivations where all units are preserved. The general idea is to ‘hide’ the disappearing unit inside of the atoms of  $A$ . By this process, subatomic proofs can be made fully linear, while in their interpretation we may observe a unit being eliminated or introduced through equations. More abstractly, we could represent subatomic proofs as a collection of strings representing the units, contained on either side of different logical relations. Each rule would simply change which logical relation each string is in the scope of, but the number of strings will remain unchanged. This work in progress is expected to shed some light on the nature of units, as well as possibly allowing for a graphical representation of subatomic proofs via strings.

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