Subatomic Proof Systems

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The proof theoretic methodology of deep inference [2] yields the widest range of analytic proof systems. In particular, several logics for which there are no analytic proof systems in Gentzen, or for which there only are cumbersome ones, admit elegant and regular analytic proof systems in deep inference. The regularity of inference rule schemes in deep inference stems from their ability (not available in Gentzen) to access the atoms which compose formulae.

The ongoing investigation on which we report here is based on the following stunning observation. If we allow inference rules to see even deeper, *inside* atoms, then we are able to reduce disparate rules such as contraction, cut, identity and any logical rule like conjunction-introduction, into a *unique* rule scheme. The very first test of this observation has to be its impact on cut elimination. We expect a simplification of the cut elimination procedure, which notoriously is determined by the shape of the inference rules. In addition, we would like to understand *why* cut elimination works on such a wide range of proof systems. We have long suspected that cut elimination is a simple and deep combinatorial phenomenon, for which we get a distorted view due to the many artifacts that proof systems typically introduce. Our preliminary results indicate that the 'subatomic' approach can be revealing because it minimises the artifacts.

Deep inference stipulates that proofs can be composed by the same connectives used to compose formulae [3,4]. For example, if $\phi = \begin{array}{c} A \\ \| \\ B \end{array}$ and $\psi = \begin{array}{c} C \\ \| \\ D \end{array}$ are two proofs whose premisses are A and C and conclusions are B and D, then $\phi \wedge \psi = \begin{array}{c} A \wedge C \\ \| \\ B \wedge D \end{array}$ and $\phi \vee \psi = \begin{array}{c} A \vee C \\ \| \\ B \vee D \end{array}$ are valid proofs with premisses $A \wedge C$ and $A \vee C$, and conclusions $B \wedge D$ and $B \vee D$. Significantly, while $\phi \wedge \psi$ can be represented in Gentzen, $\phi \vee \psi$ cannot. This is basically the definition of deep inference and it holds for every language, not just for propositional classical logic. It turns out that, as a nontrivial but direct result of this stipulation, every contraction and cut instances can be locally transformed into their atomic variants by a local procedure of polynomial-size complexity [1]. A further advantage of deep inference is that, contrary to Gentzen theory, self-dual noncommutative connectives can easily be expressed into proof systems enjoying cut elimination [5].

The main idea of this work is to consider atoms as self-dual, noncommutative binary logical relations and to build formulae by freely composing units by atoms, disjunction and conjunction.

One can quickly grasp the main idea by considering the occurrences of an atom a as interpretations of more primitive expressions involving a noncommutative binary relation, still denoted by a. Two formulae A and B in the relation a, in this order, are denoted by $A\ a\ B$. We have an enumerable supply of atoms, denoted by lowercase Latin letters, and formulae are built over the two units for disjunction and conjunction, respectively 0 and 1. For example, the two expressions $(0\ a\ 1) \vee (1\ a\ 0)$ and $(0\ b\ 1)\ a\ (1\ c\ (1\ d\ 0)) \wedge 0 \wedge (0\ a\ 0 \vee 1\ b\ 1)$ are formulae. We call *tame* the formulae where atoms do not appear in the scope of other atoms, such as the first formula, and *wild* the others, such as the second formula.

We then map tame formulae to ordinary formulae such that $0\ a\ 1 \mapsto a$ and $1\ a\ 0 \mapsto \bar{a}$, where $\bar{a}$ denotes the negation of a. We then stipulate that $0\ a\ 0 \mapsto 0$ and $1\ a\ 1 \mapsto 1$. Note that self-duality, *i.e.*, $\overline{A\ a\ B} \equiv \bar{A}\ a\ \bar{B}$, and noncommutativity, *i.e.*, $A\ a\ B \not\equiv B\ a\ A$ whenever $A \not\equiv B$, are coherent with the interpretation. We extend the interpretation $\mapsto$ to all the tame formulae in the natural way.

Let us now consider, for example, the usual contraction rule for an atom: $\dfrac{a \vee a}{a}$. We can obtain this rule as the result of applying $\mapsto$ to the formulae of some proof system where the following inference rule instance is expressed, as in $\dfrac{(0\ a\ 1) \vee (0\ a\ 1)}{(0 \vee 0)\ a\ (1 \vee 1)} \mapsto \dfrac{a \vee a}{a}$.

Surprisingly, *all* the rules needed for a complete system for many propositional logics (including classical and linear ones) are special cases of the linear rule

$$\frac{(A\ \alpha\ C)\ \beta\ (B\ \gamma\ D)}{(A\ \epsilon\ B)\ \zeta\ (C\ \eta\ D)} \quad ,$$

where the Greek letters denote logical connectives subject to certain simple conditions. We are currently in the final stages of proving a general cut elimination theorem, providing a normalisation theory to all the logics captured by those conditions.

References

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