ON ANALYTIC INFERENCE RULES IN THE CALCULUS OF STRUCTURES

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In this note, we discuss the notion of analytic inference rule for propositional logics in the calculus of structures (CoS) [4]. CoS generalises the sequent calculus and preserves all its proof-theoretic properties. There is no established notion of analytic rule outside of the sequent calculus, and so we investigate what such a notion could be for CoS.

In [3]–Section 6.4, we write that it is easy to prove that an 'analytic' restriction of CoS p-simulates CoS, and so Frege systems and sequent-calculus systems (including non-analytic ones). This could be done by adopting the following, natural notion of analyticity: a rule is analytic if, given an instance of its conclusion, the set of possible instances of the premiss is finite. If we adopted this definition, the finitary version of the atomic cut rule, called $\mathsf{fai}\uparrow$ (see Remark 6), would be analytic, and replacing the generic cut rule with $\mathsf{fai}\uparrow$ would only entail a linear cost for the size of proofs.

However, it seems like that notion of analyticity, while natural, does not lead to expected results, and it does not seem refined enough to bring to light interesting problems. So, we briefly discuss here a definition of analyticity that seems to us intuitive, that respects analyticity in the sequent calculus, and that discriminates the atomic finitary cut from the usual analytic rules.

We assume the usual basic definitions, and we refer to the introductory parts of [3]. In the following definition, we look at the set P of premisses for a given conclusion; then we make a distinction between 1) the case in which P is finite, and 2) the more restricted case in which, in addition, the cardinality $|P|$ of P does not grow as a function of the context in which the rule is applied. We say that a rule is analytic only in case 2.

Definition 1. A CoS (*inference*) *rule* is a polynomial-time computable binary relation over the set of formulae; by writing

$$r\,\frac{A}{B}$$

we indicate that formulae A (*premiss*) and B (*conclusion*) are in the inference-rule relation r. *Deep-inference* rules are the rules such that, for every context $K\{\ \}$ where the hole occurs positively,

$$\text{if} \qquad r\,\frac{A}{B} \qquad \text{then} \qquad r\,\frac{K\{A\}}{K\{B\}} \quad .$$

CoS *derivations* are sequences of formulae, such that contiguous formulae are in some inference rule. For every rule r and formula B, we define the set of *premisses of* B *in* $K\{\ \}$ *via* r:

$$\mathsf{pr}(B,K\{\ \},r) = \left\{ A \;\middle|\; r\,\frac{K\{A\}}{K\{B\}} \right\} \quad .$$

Let a rule r be given:

(1) if, for every B and $K\{\ \}$, the set $\mathsf{pr}(B,K\{\ \},r)$ is finite, then we say that r is *finitely generating*;
(2) if, for every B, there is a natural number n such that for every context $K\{\ \}$ we have $|\mathsf{pr}(B,K\{\ \},r)| < n$, then we say that r is *analytic*.

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Remark 2. An analytic rule is finitely generating.

Remark 3. According to Definition 1, all rules in the 'down' fragments of the usual CoS systems are analytic, and so are some 'up' rules, like cocontraction. The rules cut and coweakening, instead, are not finitely generating (and so, not analytic).

Remark 4. As defined in [3], the inference rule $=$ of system SKS is not analytic, because an arbitrary number of units can occur in the premiss, for any given conclusion. However, we can ask for the rule $=$ to apply only to unit-canonical formulae, where unit-canonical formulae are defined as those formulae with minimum number of units in each equivalence class modulo $=$. This way, the $=$ rule would be analytic. Another possibility is not to close the equivalence $=$ for transitivity, and allow for an arbitrary number of $=$ rules between non-$=$ inference steps, in a derivation. Both solutions do not alter any of the properties of the CoS systems involved, and they can be applied to all CoS systems where equations are used.

Remark 5. We can apply Definition 1 to the natural translation of the sequent calculus rules into CoS. As desired, all inference rules of a standard sequent-calculus system are analytic, except for the cut rule, which is not finitely generating. On the contrary, and as desired, Smullyan's analytic cut rule [5] is analytic according to Definition 1.

Remark 6. The *atomic finitary cut* rule

$$\mathsf{fai}\!\uparrow \frac{K\{a \wedge \bar{a}\}}{K\{\mathsf{f}\}} \quad , \qquad \text{where } a \text{ appears in } K\{\ \} \quad ,$$

is finitely generating but not analytic, as desired. This rule is defined in [1] and briefly discussed in [3]–Section 6.4.

References

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