MISMATCH

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There is a mismatch between meta and object levels in many calculi, like the sequent calculus and natural deduction. The mismatch has undesirable proof theoretical consequences, the most important being the inability to design deductive systems. Since the object level is untouchable (it's the language you want to deal with), one can fix the problem by \`improving´ the meta level; or one can also sort of *merge* object and meta level by using deep inference.

1 There Is a Mismatch Between Meta and Object Level in Many Calculi

I will here make more precise what I mean by \`mismatch´. It's not worth the effort to make this notion formal, because it has more of a moral value than a technical one. To make my point, looking at the sequent calculus suffices, because the reader can immediately make the analogue with natural deduction, proof nets, etc.

1.1 Let us first see a case of perfect match in Gentzen's one-sided sequent calculus, for propositional classical logic. Consider:

$$\wedge\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \wedge B, \Gamma, \Delta} \quad , \qquad \vee\frac{\vdash A, B, \Gamma}{\vdash A \vee B, \Gamma} \quad ,$$

where A and B are formulae and Γ and Δ are multisets of formulae. In the first inference, the object level $\wedge$ in the conclusion corresponds to a meta level \`and´ between the two premises (branches of the derivation). In the second inference, the object level $\vee$ corresponds to the meta level \`or´ expressed by commas in the premise sequent.

We can test the correspondence by \`flattening down to the object level´ the meta level and check whether inferences correspond to implications. They do, because

$$((A \vee \Gamma) \wedge (B \vee \Delta)) \Rightarrow ((A \wedge B) \vee \Gamma \vee \Delta) \quad ,$$

and trivially so for the $\vee$ rule.

There is no mismatch, because, when going bottom-up, the connectives at the object level ($\wedge$, $\vee$) become their corresponding structural relations at the meta level (resp. \`branching´ and \`being in a multiset´).

1.2 Let us now see a case of mismatch, in Gentzen's one-sided sequent calculus for linear logic. Consider:

$$\otimes \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} ,$$

where ⊗ is multiplicative conjunction (\`times´ or \`tensor´). Can we consider branching a meta-level ⊗? We could, because

$$((A \# \Gamma) \otimes (B \# \Delta)) \multimap ((A \otimes B) \# \Gamma \# \Delta) ,$$

where # is multiplicative disjunction (\`par´) and −o is linear implication.

Now let's see additive conjunction & (\`with´):

$$\& \frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \& B, \Gamma} .$$

We assumed that branching is ⊗, and we find that

$$((A \# \Gamma) \otimes (B \# \Gamma)) \not\multimap ((A \& B) \# \Gamma) ,$$

so we have a mismatch. If we assume that branching is &, instead of ⊗, then

$$((A \# \Gamma) \& (B \# \Gamma)) \multimap ((A \& B) \# \Gamma) ,$$

so the mismatch disappears here, but it reappears in the previous case of ⊗, because

$$((A \# \Gamma) \& (B \# \Delta)) \not\multimap ((A \otimes B) \# \Gamma \# \Delta) .$$

We could go one step further by declaring that branching between sequents (or derivations) S and T corresponds to !S ⊗ !T, where ! is the \`of course´ modality. In this case, we have

$$(!(A \# \Gamma) \otimes !(B \# \Delta)) \multimap ((A \otimes B) \# \Gamma \# \Delta) \quad \text{and}$$
$$(!(A \# \Gamma) \otimes !(B \# \Gamma)) \multimap ((A \& B) \# \Gamma) .$$

But are we happy? No, because we still have a mismatch, since !(.) ⊗ !(.) (i.e., the branching relation) doesn't correspond neither to ⊗, nor to &, nor to any other connective in the language.

At this point you see how I could technically define my notion of mismatch, in the sequent calculus. Of course, the notion would be meaningful also in proof nets, but the definition for them would become more delicate (given that branching is rather different), so it's not worthy to get into this business for now.

What matters is that, for example, if one asks the question: \`What does branching correspond to?´ then the answer cannot be given simply and elegantly in terms of logical operators at the object level. I'm going to argue that this difficulty causes trouble in structural proof theory.

2 The Mismatch Has Undesirable Proof Theoretic Consequences

My impression is that we can consider the sequent calculus `perfect´ for classical logic, since there's no mismatch, but it gets less and less adequate the more we depart from classical logic.

2.1 If we move to linear logic, we see disturbing phenomena occurring already, even more disturbing than the mismatch observed above. Consider for example the promotion rule:

$$\mathsf{p}\,\frac{\vdash A,\ ?B_1,\ \ldots,\ ?B_h}{\vdash\ !A,\ ?B_1,\ \ldots,\ ?B_h}\ ,$$

where ? (`why not´) is the dual modality of !. Contrary to legitimate expectation, applying the rule requires a global check on the entire context of !A, for making sure that all formulae are modalised by ?.

If one is only interested in certain technical aspects of deductive systems, like for example cut admissibility, then the sequent calculus, in this case, does its job, and we only have to pay a price in terms of elegance and conceptual clarity. In this sense, the rule above is disturbing, for example, because the `meaning´ of ! requires a side condition on the context. As we will see later, pure aesthetics of this rule can also be improved in a different approach.

The promotion rule is not `difficult´ only in the sequent calculus: proof nets for ! and ? are also unsatisfying, again for problems that one can ascribe to the mismatch.

2.2 What I find the most serious drawback of the mismatch is the impossibility, or at least the inability, or at the very least the difficulty, of designing deductive systems for many logics that are otherwise known:

- Impossibility: we know a logic, called BV, for which Alwen Tiu proved that it's impossible to express it in Gentzen's sequent calculus [AT]. It's the only result I know of this kind. There is no room here to give the details, suffices to say that BV is extremely simple, and it's not a specially tailored pathological case.

- Inability and difficulty: probably the most common cases of inability are found in modal logics. For several modal logics, like S5, it is not known (if I'm not wrong) whether a (pure) Gentzen's sequent calculus presentation is achievable. For other modal logics one such presentation has been painfully found, often after significant departures from the pure form of Gentzen's sequents.

Typically, the context structure of sequents is enriched: they are no more just sets or multisets, but become order structures of varying, and sometimes very high, complexity. All of this is done of course for the sake of eliminating the mismatch.

2.3 There is a simple philosophy of Gentzen's sequent calculus, for which, grossly speaking, the `meaning' of connectives is defined by their rules, where connectives are `brought to the meta level´ with no reference to the context, according to some harmony-preserving principles. Departing from pure sequents has a conceptual cost, which is not entirely repaid, in my opinion, by achieving technical results like cut elimination.

In fact, there is a value in having simple deductive systems, all designed according to the same methodology. Having to tailor the shape of sequents to each logic requires an effort, both for the proof theorist and for the user of the deductive system, who has to learn a new metalanguage for every system.

There is in addition a technological cost. `Exotic´ logics are more and more used in computer science, in verification, for example. If one changes the metalanguage (of sequents, or of natural deduction, typically), then the corresponding verification system must be updated accordingly. In other words, it is difficult to design a universal framework in which all sorts of metalanguages for describing deductive systems are easily coded.

3 One Can Fix the Problem by `Improving´ the Meta Level

As I said above, the standard way of dealing with the mismatch is to enrich the structure of the meta level. The idea is simple: if branching and commas are not enough, let's supply some more structure.

Since doing this ad hoc, case by case, blurs the conceptual picture and has costs for the users of deductive systems, a general solution is desirable.

As far as I know one strong bid in this direction comes from Belnap's display calculus [NB]. The idea goes as follows:

3.1 Why does one need a meta level at all? Because, when looking for a proof, one has to get inside formulae to access the information lying there under several strata of connectives. To do so, one breaks formulae into pieces, but has to keep track carefully of the relations between pieces, whence the need of structure at the meta level.

The display calculus does this: it defines a rich notion of structure at the meta level, and makes sure that the various subformulae are accessible (`on display´). Then a general cut elimination theorem exists which is automatically inherited by all deductive systems presented in the display calculus. Since the display calculus is mainly studied by philosophers, I'm sure that a careful theory of meaning has been elaborated (I'm not knowledgeable enough to be more positive).

3.2 I'd be very much interested in listing all advantages and disadvantages of the display calculus. My provisory list goes as follows:

- Advantages: uniformity, for example many modal logics receive a uniform treatment here, as opposed to ad hoc presentations in variants of Gentzen's sequent calculus; generality, one cut elimination theorem suffices for all systems.

- Disadvantages: complexity, I find the definition of display calculus `very big´.

As a last remark, I have to say that I'm not entirely convinced of the display calculus approach because it goes in the direction of complexity instead of simplicity. There are two ways to reach generality: 1) to take into account all possibilities and then study them once and for all, 2) to try and find a simple principle underlying a vast class of phenomena. My impression is that the display calculus does 1, while I believe that in structural proof theory something like 2 can be discovered.

4 One Can `Bring the Meta Level Down to the Object One´ by Using Deep Inference

If the purpose of the meta level is to make subformulae accessible, why don't we just get in and take them, by inferring directly inside formulae, at any level of depth? Of course, one reason not to do so could be being unable to do proof theory (for example proving cut elimination)!

My colleagues and I, in the last few years, have developed a proof theory for deep inference (called the *calculus of structures*), and we claim to have a radical, good solution for the problem represented by the mismatch. I will explain here the main idea with a couple of examples, and will make a summary of the advantages I see in my approach.

4.1 All inference rules are of the form

$$\frac{C\{B\}}{C\{A\}} \;,$$

meaning that subformula A inside formula C{A} is replaced inside context C{ } by B, while going up in building a proof. There is no branching, no commas, no meta-level structure. The formula itself takes care of representing what used to be the meta-level structure. Given this, rule $\vee$ of classical logic immediately disappears. Let us just assume from now on that A and B, as above, do not occur in the scope of an odd number of negations (otherwise we could use dual rules to the ones we are going to see).

Rule $\wedge$ becomes what I call a `switch´:

$$\mathrm{s}\ \frac{C\{(A \vee D) \wedge (B \vee E)\}}{C\{(A \wedge B) \vee (D \vee E)\}}\ ,$$

which is good also for the linear logic case,

$$\mathrm{s}\ \frac{C\{(A \# D) \otimes (B \# E)\}}{C\{(A \otimes B) \# (D \# E)\}}\ .$$

4.2 This was of course a triviality. The next step is looking at cut and cut elimination. Identity and cut look like this:

$$\mathrm{id}\ \frac{C\{\mathit{true}\}}{C\{A \vee \neg A\}} \quad \text{and} \quad \mathrm{cut}\ \frac{C\{A \wedge \neg A\}}{C\{\mathit{false}\}}\ .$$

It is easy to see how they correspond to their counterparts in the sequent calculus. What is not easy is proving cut elimination, because, in the presence of deep inference, the old methods are basically useless. One can find in [KB1, KB2, LS1] proofs of cut elimination for classical and linear logics. Many other papers are available at [WS], one of which also deals with a uniform theory for modal logics [SS].

4.3 Rules like promotion get important advantages from deep inference. Here is promotion in the calculus of structures:

$$\mathrm{p}\ \frac{C\{!(A \# B)\}}{C\{!A \# ?B\}}\ ;$$

since there's more flexibility in managing the context than just branching, this promotion rule can check its context incrementally, one ?B at a time, this way removing the need for the global test that afflicted promotion in the sequent calculus. (Why this is so is not obvious, the reader can check [LS1] or [GS] or [LS2] for details.)

More importantly, one can see that promotion is just a unary case of switch. As a matter of fact, rules in the calculus of structures can be derived from a more general, simple scheme, briefly described in [AG1]. (An even more dramatic simplification is speculated in [AG2].)

4.4 The biggest advantage that deep inference brings into the picture is top-down symmetry for derivations. A derivation can be flipped upside-down, and negated, and it remains a valid derivation. One can see our new symmetry in the identity and cut rules shown above. In the sequent calculus, this is impossible because there is a top-down asymmetry: while going up, one has less and less of object level and more and more of meta level.

The top-down symmetry has important proof theoretic consequences. For example, the cut rule can be equivalently replaced by its atomic version (the principal formula is an atom). This can be done without having to prove full-blown cut elimination, but just by a very simple inductive argument dual to the corresponding one for identity. This contributes to completely different, and in certain cases very simple, arguments for cut elimination (see [KB1]). In general, the entire analysis of proofs benefits from the symmetry, for example several interesting notions of normal forms can be studied which are provably impossible in the sequent calculus.

4.5 In conclusion, given that the mismatch causes trouble, it seems to me that the solution of eliminating the cause of the disease, by deep inference, looks better than the alternative of living with it by curing the symptoms. There are (at least) two serious objections one can make:

1 There is no \`theory of meaning´ for what we do with deep inference. My guess is that this might come one day, given that my formalism is very clean and regular, and it possesses interesting properties, that should stimulate the interest in studying it from a philosophical perspective.

2 There is no general cut elimination theorem, like for display logic. Again, I believe that this is a matter of time: I am especially confident in the kind of research started in [AG2] and in so-called splitting theorems (see [AG3] for an example), which appear to have the right kind of encompassing regularity needed for this task.

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