# ON LAFONT'S COUNTEREXAMPLE

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Lafont's counterexample shows why cut elimination in the sequent calculus of classical logic is not confluent. I show here that the counterexample doesn't work in the calculus of structures [WS].

## Lafont's Counterexample

Take two proofs,

```
 ------          ------
  \Π1/            \Π2/
   \/      and     \/    ,
  |- A            |- B
```

and combine them as in

```
    ------      ------
     \Π1/        \Π2/
      \/          \/
     |- A        |- B
  w-------    w--------
    |- A, C     |- A, ¬C
cut-------------------- .
          |- A, B
```

When you perform cut elimination you only have a choice between

```
  ------           ------
   \Π1/             \Π2/
    \/               \/
   |- A             |- B
w-------   and   w------- .
 |- A, B          |- A, B
```

Since $\Pi_1$ and $\Pi_2$ can be totally unrelated, confluence is lost. A slightly different situation, and some discussion, is in [GLT].

## The Problem

The usual sequent systems don't allow to keep *both* $\Pi_1$ *and* $\Pi_2$ when proving (|- A, B). Proving (|- A, B) means proving A $\vee$ B; of course it's enough proving A (or proving B), but it wouldn't be wrong to prove them both and then collect both proofs. If we keep this information, we can hope to get meaningful denotations.

As is known, the problem disappears if we *add* a mix rule to the sequent calculus: we could just consider as the contractum

```
      ------    ------
       \Π1/      \Π2/
        \/        \/
       |- A      |- B
   mix------------- .
          |- A, B
```

## What Happens in the Calculus of Structures

In system SKS [BT], the job of mix is done by the *already present* switch rule, together with units. Take

```
         ___
    Π1◊Π2|
      (A,[f,B]) = (A,B) ≈ A ∧ B
   w↓---------
      (A,[t,B])
    s---------                   ,
      [(A,t),B] = [A,B] ≈ A ∨ B
```

where **f** and **t** are *false* and *true* and $\Pi_1 \lozenge \Pi_2$ stands for any interleaving of the two proofs $\Pi_1$ and $\Pi_2$.

There are many possible ways of interleaving $\Pi_1$ and $\Pi_2$, and there are two ways of introducing the **t** unit (by A, as above, or by B). This means that we cannot get confluence *directly* in the calculus of structures, but, for example, we have to resort to related proof nets, which are in the works.

The point, though, is that all the information needed for a denotation is *naturally* retained, and it looks feasible to discard the irrelevant ordering of structures and popping up of units.

## References


[BT] Kai Brünnler and Alwen Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347-361. Springer-Verlag, 2001. URL: http://www.ki.inf.tu-dresden.de/~kai/LocalClassicalLogic-lpar.pdf.

[GLT] Jean-Yves Girard, Yves Lafont and Paul Taylor. *Proofs and types*. Cambridge University Press, 1989.


## Web Site


[WS] http://www.ki.inf.tu-dresden.de/~guglielm/Research.