RED AND BLUE

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This note needs to be read in colour to be meaningful!

In the sequent calculus, if we paint $\wedge$ in red and blue and use the same rules for differently coloured $\wedge$s, then we can prove the equivalence of red and blue $\wedge$s. The same is true for $\vee$ and many connectives, for example in linear logic it is true for all connectives but not for the modalities ! and ?. We argue here that this is not satisfactory of the sequent calculus (SC), while in the calculus of structures (CoS) the situation can be mastered satisfactorily.

Red and Blue Connectives in the Sequent Calculus

Take a SC system for classical logic, and suppose that conjunction and disjunction come in two colours, red and blue. We have then two connectives $\color{red}{\wedge}$ and $\color{blue}{\wedge}$ (and the same for disjunction), and we suppose that the same rules of inference apply on both. We can then prove that $F \color{red}{\wedge} \color{black}{G} \Leftrightarrow F \color{blue}{\wedge} \color{black}{G}$:

$$\color{red}{\wedge}_L\,\color{black}{\dfrac{\color{blue}{\wedge}_R\,\color{black}{\dfrac{\dfrac{}{F, G \vdash F} \qquad \dfrac{}{F, G \vdash G}}{F, G \vdash F \color{blue}{\wedge} \color{black}{G}}}}{F \color{red}{\wedge} \color{black}{G \vdash F} \color{blue}{\wedge} \color{black}{G}}} \qquad \text{and} \qquad \color{blue}{\wedge}_L\,\color{black}{\dfrac{\color{red}{\wedge}_R\,\color{black}{\dfrac{\dfrac{}{F, G \vdash F} \qquad \dfrac{}{F, G \vdash G}}{F, G \vdash F \color{red}{\wedge} \color{black}{G}}}}{F \color{blue}{\wedge} \color{black}{G \vdash F} \color{red}{\wedge} \color{black}{G}}} \quad .$$

My first impression was that the SC does the right thing, because if two connectives behave the same way, they should be indistinguishable.

Why do we get the equivalence in the SC? Because in the SC both the red and blue connectives are reduced to the same meta-level structures by the inference rules. In the example above, both the $\color{red}{\wedge}$ in the left proof and the $\color{blue}{\wedge}$ in the right proof are reduced to indistinguishable commas; analogously, the $\color{blue}{\wedge}$ in the left proof and the $\color{red}{\wedge}$ in the right proof both give rise to (the same kind of) branching. This implicit common reduction is crucial for getting the equivalence.

In fact, when this does not happen, there is no equivalence. Such is the case for modalities in linear logic: we can consider $\color{red}{!}$ and $\color{blue}{!}$ in linear logic, and we soon discover that they are not equivalent (in the sense of $\color{red}{!}\color{black}{F} \multimap\!\!\circ\ \color{blue}{!}\color{black}{F}$). The reason is that the rules for modalities make no reference to a *common* meta level; consider, for example, red promotion:

```
    !Γ |- F, ?Δ
pr ------------ .
    !Γ |- !F, ?Δ
```

The contexts !Γ and ?Δ have the same colour of the modality the rule acts upon and are not reduced to the meta level.

In my opinion, this obviously disturbing situation has nothing to do with the logic (in this case linear logic), but only exposes the bias of the SC for classical logic: the meta-level of the SC is intrinsically classical and has difficulties in coping with different logics. This is not the case for CoS, where the meta level does not exist, what makes the formalism agnostic regarding the logics expressed in it.

Red and Blue Connectives in the Calculus of Structures

The obvious system for classical logic with red and blue disjunction and conjunction in CoS would be the following modification of system KSg (see [KB]), let's call it KSg´:

```
       t              (R [T U])         f         [R R]
i↓ -------- ,   s↓ ----------- ,   w↓ --- ,   c↓ ------- ,
    [R -R]            [(R T) U]         R           R

       t              (R [T U])         f         [R R]
i↓ -------- ,   s↓ ----------- ,   w↓ --- ,   c↓ ------- .
    [R -R]            [(R T) U]         R           R
```

Here, the red and blue world are completely separate, the rules only put in relation connectives of the same colour. In order to prove the equivalence of red and blue connectives, we should prove the following four structures in KSg´:

```
[(R T) [-R -T]] ,   [(R T) [-R -T]] ,

[(R T) [-R -T]] ,   [(R T) [-R -T]] .
```

Notice that we cannot appeal here to a common notion of implication (like we could do in the sequent calculus), rather there are two, a red one and a blue one. None of the structures above is provable in the given system.

At first, we might not be happy about this, because, as said above, `if the connectives behave the same then they must *be* the same´. However, notice that the behaviour, as specified in CoS, is always *relative* of a connective to another one, i.e., there is no meta level to confront with, and no induced notion of implication (the turnstile in the SC). So, it's not obvious what `behaving the same´ could mean, it's something one has to be very precise about.

As a matter of fact, if we do want a logic with two couples of coloured connectives and some interaction between them, there are unlimited possibilities in CoS. For example, we might consider adding the following two rules to KSg´:

$$s'{\downarrow}\;\frac{(R\ [T\ U])}{[(R\ T)\ U]}\ ,\qquad s'{\downarrow}\;\frac{(R\ [T\ U])}{[(R\ T)\ U]}\ .$$

By doing this, red and blue would still `behave the same´, and we would get another, more `interactive´, notion of implication which is intermediate between that of complete independence and equivalence (notice that these two rules still don’t allow to prove equivalence as stated above).

It is certainly possible to concoct something similar in the SC, for example by painting the commas in red and blue and designing rules which also take their colour into account. However, it is awkward to distinguish red branching from blue branching!

Conclusion

We are dealing here with yet another artifact of the SC. The behaviour of differently painted connectives induces equivalence in the SC only when the connectives happen to coincide enough with their meta-level counterparts. In CoS, and in general in deep inference, there are no constraints and one is free to induce equivalence or not.

References

[KB] Kai Brünnler. *Deep Inference and Symmetry in Classical Proofs*. Logos Verlag, Berlin, 2004. URL: http://www.iam.unibe.ch/~kai/Papers/phd.pdf.

Web Site

http://alessio.guglielmi.name/res/cos.