GOODNESS, PERFECTION AND MIRACLES

Alessio Guglielmi (TU Dresden)
15.11.2000 - updated on 18.10.2002

In this note I introduce three properties closely related to cut elimination and interpolation theorems. They find very natural definitions in the calculus of structures [WS].

1 Definition A cut-free formal system S in the calculus of structures is *good* if whenever [-R,T] is provable then there is a derivation

```
R
|S ,
T
```

under the following constraints:
1 occ(R) = occ(T);
2 there are no dual atoms in occ(R) (and so in occ(T)).
(The set occ(R) contains all atom occurrences of R.)

Goodness is then sort of a \`deductive completeness´ property. In the calculus of structures it is in general easy to get a good system: just add rules until you get goodness. But then the system would not necessarily be perfect:

2 Definition A formal system S is *perfect* if it is good and there are no admissible rules, except possibly for atomic cut.

I find perfection a very appealing notion, because it makes for a *minimalist* matching of derivability and implication. On the other hand, it requires also an asymmetric constraint due to (absence of) admissibility, and this could prove too ambitious.

3 Definition Let R and T be two structures such that occ(R) = λ + μ and occ(T) = λ + ν, where μ and ν are disjoint. We say that S enjoys the *miracle property* if, whenever there is a derivation

```
 R
d|S
 T
```

then there is a derivation

```
  R
d'|S
  T
```

such that the only occurrences of i↓ rules operate on atoms of ν and the only occurrences of i↑ rules operate on atoms of μ. (It is equivalent using ai↓ and ai↑ rules instead of i↓ and i↑.)

In other words: Some of the atom occurrences in R and T are in common, and they belong to the set λ. Some others are not, and they belong to disjoint sets μ and ν. The atoms in μ appear in R and not in T, so they have to be introduced going up, in cut rules; the atoms in ν appear in T and not in R, so they have to be introduced going down, in identity rules. The question is whether being able to introduce only the atoms in ν and μ is enough. If so, then the miracle property holds, and it trivially entails cut-elimination and goodness.

The above definitions are appropriate for linear systems, i.e., systems where no contraction and weakening rules apply. For systems with contraction and/or weakening, it's more appropriate to replace, in the above definitions, the set of occurrences occ(R) by the set at(R) of atoms appearing in R. I will implicitly do so in the following.

SBV is the commutative/non-commutative linear logic studied in [AG]; system BV is the down fragment of SBV.

We can state the following:

4 Fact *System BV is not good.*

Proof $[-(R,T),\langle R;T\rangle] = [-R,-T,\langle R;T\rangle]$ is provable, but the only derivation of $\langle R;T\rangle$ from (R,T) needs the admissible rule $q\uparrow$, which belongs to SBV, but not to BV:

$$q\uparrow\frac{(R,T)}{\langle R;T\rangle}\ .$$

QED

SBV could be good, but it is not perfect, since $q\uparrow$ is admissible:

5 Conjecture *System SBV is good.*

We can perhaps get a stronger result:

6 Conjecture *System SBV enjoys the miracle property.*

But is there any perfect system? I believe so: FBV is the \`flat´ version of system BV; FBV is equivalent to multiplicative linear logic plus mix and nullary mix. The following should be easily provable:

7 Conjecture *System FBV is perfect and FBV with atomic cut enjoys the miracle property.*

Of course, FBV is a very weak system, so the question remains whether perfection can be obtained for more expressive systems. The situation might improve (with respect to SBV) for the systems available for classical logic, in particular SKS and its down fragment KS [BT].

8 Conjecture *System KS is perfect.*

9 Conjecture *Systems SKS enjoys the miracle property.*

Proving this conjecture should lead to an interpolation theorem *on derivations*, in addition to providing cut elimination as a trivial corollary.

What about linear logic?

10 Problem *Does some system for linear logic in the calculus of structures enjoy the miracle property? And perfection?*

Goodness doesn't hold in the systems available now, so neither the miracle property nor perfection can hold for them. Take for example system SELS [GS], which is a presentation of MELL (the multiplicative exponential fragment of linear logic) in the calculus of structures.

11 Fact *SELS is not good.*

Proof [-?[R,T],?R,?T] = [!(-R,-T),?R,?T] is provable but there is no derivation in SELS for

```
?[R,T]
   |    .
[?R,?T]
```

QED

As I said, goodness is easy to get: for example, simply allow the derivation above by adding it as a rule. This would make looking for the miracle property possible, but this solution is somewhat inelegant; perhaps there's some better way.

(We can get a more technical terminology by calling *implicationally complete* a good system, and *structurally complete* a system which enjoys the miracle property.)

References

[BT] Kai Brünnler and Alwen Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347-361. Springer-Verlag, 2001. URL: http://www.ki.inf.tu-dresden.de/~kai/LocalClassicalLogic-lpar.pdf.

[AG] Alessio Guglielmi. A system of interaction and structure. Technical Report WV-02-10, Dresden University of Technology, 2002, URL: http://www.ki.inf.tu-dresden.de/~guglielm/Research/Gug/Gug.pdf, submitted.

[GS] Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In L. Fribourg, editor, *CSL 2001*, volume 2142 of *Lecture Notes in Computer Science*, pages 54-68. Springer-Verlag, 2001. URL: http://www.ki.inf.tu-dresden.de/~guglielm/Research/GugStra/GugStra.pdf.

Web Site

[WS] http://www.ki.inf.tu-dresden.de/~guglielm/Research.