A Local System for Classical Logic

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Abstract. The *calculus of structures* is a framework for specifying logical systems, which is similar to the one-sided sequent calculus but more general. We present a system of inference rules for propositional classical logic in this new framework and prove cut elimination for it. The system enjoys a decomposition theorem for derivations that is not available in the sequent calculus. The main novelty of our system is that all the rules are *local*: contraction, in particular, is reduced to atomic form. This should be interesting for distributed proof-search and also for complexity theory, since the computational cost of applying each rule is bounded.

1 Introduction

When implementing inference systems, in a distributed fashion especially, the need to copy formulae of unbounded size is generally considered problematic. In the sequent calculus, it is caused by the contraction rule, e.g. in Gentzen's LK [2]:

$$\frac{\mathit{\Gamma} \vdash \mathit{\Phi}, A, A}{\mathit{\Gamma} \vdash \mathit{\Phi}, A} \quad .$$

Here, going from bottom to top in constructing a proof, a formula A of unbounded size is duplicated. Whatever mechanism performs this duplication, it has to inspect all of A, so it has to have a *global* view on A. While this can be taken for granted on a single processor system, it is harder to achieve on a distributed system, where each processor has a limited amount of local memory. The formula A could be spread over a number of processors. In that case, no single processor has a global view on A.

Let us call *local* those inference rules that do not require such a global view on formulae of unbounded size, and *non-local* those rules that do. Besides contraction, another example of clearly non-local behaviour is provided by the promotion rule in the sequent calculus for linear logic [3]. To remove an exclamation mark from one formula, it has to check whether all formulae in the context are prefixed with a question mark. The number of formulae to check is unbounded:

$$\frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n} \quad .$$

While there are methods to solve these problems in the implementation, an interesting question is whether it is possible to solve them proof-theoretically, i.e. by avoiding non-local rules altogether. This question is answered positively in this paper for the case of classical propositional logic. The predicative case is work in progress and is sketched in the conclusion.

Locality is achieved by reducing the problematic rules to their atomic forms. This is not entirely new: there are sequent systems for classical logic in which the identity axiom is reduced to its atomic form, i.e.

$$\overline{A \vdash A} \qquad \text{is admissible for} \qquad \overline{a \vdash a} \quad ,$$

where a is an atom. However, we do not know of any sequent system in which contraction and weakening are admissible for their atomic forms. In fact, we believe that such a system does not exist. To achieve our goal, we depart from the sequent calculus and employ the recently conceived *calculus of structures* [5]. In contrast to the sequent calculus, it does not rely on the notion of main connective and permits the application of rules anywhere *deep* inside a formula, exploiting the fact that implication is closed under disjunction and conjunction. This ability is crucial for the rules of our system. The calculus of structures has already successfully been employed in [7] to solve the problem of the non-local behaviour of the promotion rule.

This paper is structured as follows: first, we introduce basic notions of the calculus of structures. Then we present our system, named SKS, and argue that its rules are local. We prove that it is equivalent to the Gentzen-Schütte formulation of classical logic, prove cut elimination and state two decomposition theorems for derivations. In the end, some open problems are identified.

2 Structures and Derivations

Definition 2.1. There are infinitely many *literals*. Literals, positive or negative, are denoted by a, b, ... There are two special literals, *true* and *false*, denoted t and f. The *structures* of the language KS are generated by

$$S ::= a \mid [\underbrace{S,\dots,S}_{>0}] \mid (\underbrace{S,\dots,S}_{>0}) \mid \bar{S} \quad ,$$

where $[S_1,\dots,S_h]$ is a *disjunction* and $(S_1,\dots,S_h)$ is a *conjunction*. $\bar{S}$ is the *negation* of the structure S. Structures are denoted by S, P, Q, R, T, U, V and W. Structures with a hole that does not appear in the scope of a negation are denoted by $S\{\ \}$. The structure R is a *substructure* of $S\{R\}$, and $S\{\ \}$ is its *context*. We simplify the indication of context in cases where structural parentheses fill the hole exactly: for example, $S[R,T]$ stands for $S\{[R,T]\}$. Structures are considered to be syntactically equivalent modulo the relation $=$, which is the smallest congruence relation induced by the equations shown in Fig. 1, where $\boldsymbol{R}$ and $\boldsymbol{T}$ stand for finite, non-empty sequences of structures.

Associativity

$$[\boldsymbol{R}, [\boldsymbol{T}]] = [\boldsymbol{R}, \boldsymbol{T}]$$
$$(\boldsymbol{R}, (\boldsymbol{T})) = (\boldsymbol{R}, \boldsymbol{T})$$

Commutativity

$$[\boldsymbol{R}, \boldsymbol{T}] = [\boldsymbol{T}, \boldsymbol{R}]$$
$$(\boldsymbol{R}, \boldsymbol{T}) = (\boldsymbol{T}, \boldsymbol{R})$$

Singleton

$$[R] = R = (R)$$

Constants

$$[\mathsf{f}, \boldsymbol{R}] = [\boldsymbol{R}]$$
$$(\mathsf{t}, \boldsymbol{R}) = (\boldsymbol{R})$$

Negation

$$\bar{\mathsf{t}} = \mathsf{f}$$
$$\bar{\mathsf{f}} = \mathsf{t}$$
$$\overline{[R_1, \dots, R_h]} = (\bar{R}_1, \dots, \bar{R}_h)$$
$$\overline{(R_1, \dots, R_h)} = [\bar{R}_1, \dots, \bar{R}_h]$$
$$\bar{\bar{R}} = R$$

Fig. 1. Syntactic equivalence on structures

Structures are somewhere between formulae and sequents. They share with formulae their tree-like shape and with sequents the built-in, decidable equivalence modulo associativity and commutativity. Structures have a normal form, unique modulo commutativity, where negation only occurs in the form of negative literals and all constants that can be removed are removed. In all inductive arguments to come, structures are considered to be in normal form.

Definition 2.2. An *inference rule* is a scheme of the kind

$$\rho \frac{S\{T\}}{S\{R\}} \quad ,$$

where ρ is the *name* of the rule, $S\{T\}$ is its *premise* and $S\{R\}$ is its *conclusion.* The context $S\{\ \}$ may be empty. In an instance of ρ, the structure taking the place of R is called *redex* and the structure taking the place of T is called *contractum.* A (*formal*) *system* $\mathscr{S}$ is a set of inference rules.

Definition 2.3. A *derivation* Δ in a certain formal system is a finite chain of instances of inference rules in the system:

$$\pi' \frac{T}{\pi \frac{V}{\frac{\vdots}{\rho' \frac{}{\rho \frac{U}{R}}}}} \quad .$$

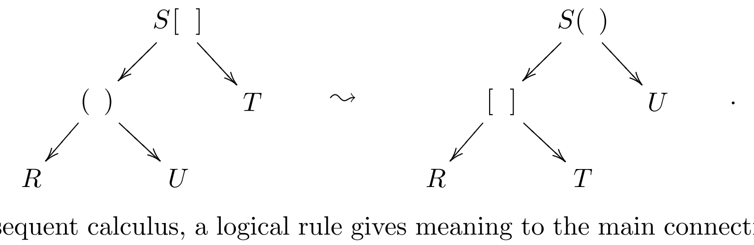
A derivation can consist of just one structure. The topmost structure in a derivation is called the *premise* of the derivation, and the structure at the bottom is called its *conclusion*. A derivation Δ whose premise is T, whose conclusion is R, and whose inference rules are in $\mathscr{S}$ will be indicated with $\begin{array}{c} T \\ \Delta \Vert \mathscr{S} \\ R \end{array}$. A *proof* Π in the calculus of structures is a derivation whose premise is t. It will be denoted by $\begin{array}{c} \Pi \Vert \mathscr{S} \\ R \end{array}$. A rule ρ is *strongly admissible* for a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every instance of $\rho\,\dfrac{T}{R}$ there is a derivation $\begin{array}{c} T \\ \Delta \Vert \mathscr{S} \\ R \end{array}$. A rule ρ *permutes over* a rule π (or π *permutes under* ρ) if for every derivation $\pi\,\dfrac{T}{\rho\,\dfrac{U}{R}}$ there is a derivation $\rho\,\dfrac{T}{\pi\,\dfrac{V}{R}}$ for some structure V.

3 System SKS

System SKS is shown in Fig. 2. The first S stands for "symmetric" or "self-dual", meaning that for each rule its dual (or contrapositive) is also in the system. The K stands for "klassisch" as in Gentzen's LK and the last S means that it is a system on structures.

The rules $\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{m}, \mathsf{aw}\!\downarrow, \mathsf{ac}\!\downarrow$ are called respectively *atomic identity*, *switch*, *medial*, *atomic weakening* and *atomic contraction*. Their dual rules carry the same name prefixed with a "co-", so e.g. $\mathsf{aw}\!\uparrow$ is called *atomic co-weakening*. The rule $\mathsf{ai}\!\uparrow$ is special, it is called *atomic cut*. Rules $\mathsf{ai}\!\downarrow, \mathsf{aw}\!\downarrow, \mathsf{ac}\!\downarrow$ are called *down-rules* and their duals are called *up-rules*.

Note that no rule requires the duplication or the comparison of structures of unbounded size. The atomic rules only need to duplicate or compare literals. The two rules that involve structures of unbounded size are m and s. Since they do not duplicate or compare the structures held by R, T, U and V, there is no need to inspect those structures at all. Consider structures represented as trees in the obvious way. Then the switch rule can be implemented by changing the marking of two nodes and exchanging two pointers (similarly for medial):



In the sequent calculus, a logical rule gives meaning to the main connective of a formula by saying that the formula is provable if certain immediate subformulae are provable. During a proof-search, formulae successively get decomposed, with their main connectives disappearing.

$$\mathsf{ai}\!\downarrow\frac{S\{\mathsf{t}\}}{S[a,\bar{a}]}\qquad\qquad \mathsf{ai}\!\uparrow\frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{m}\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\mathsf{aw}\!\downarrow\frac{S\{\mathsf{f}\}}{S\{a\}}\qquad\qquad \mathsf{aw}\!\uparrow\frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\mathsf{ac}\!\downarrow\frac{S[a,a]}{S\{a\}}\qquad\qquad \mathsf{ac}\!\uparrow\frac{S\{a\}}{S(a,a)}$$

Fig. 2. System SKS

The rules switch and medial of system SKS do not fit into this scheme. Not only are they applicable deep inside a formula (or structure, for that matter), there also is no main connective that is removed. While there is a connection between the switch rule and the R∧ rule in the sequent calculus (cf. the proof of Theorem 4.2), the medial rule bears no resemblance of any sequent calculus rule. Its premise is a disjunction and its conclusion a conjunction. This is impossible in the sequent calculus, where the conclusion of a rule is always a disjunction (a sequent) and the premise of a rule is either also a disjunction (for single premise rules) or a conjunction (for two premise rules, since the two premises are logically in a conjunction).

Remark 3.1. When talking about derivations, taking the dual means turning them upside-down, thereby exchanging premise and conclusion, and replacing each connective and constant by its De Morgan dual. For example

$$\mathsf{s}\,\frac{\mathsf{s}\,\frac{\mathsf{ai}\downarrow\,\frac{\mathsf{ai}\downarrow\,\frac{\mathsf{t}}{[b,\bar{b}]}}{([a,\bar{a}],[b,\bar{b}])}}{[b,([a,\bar{a}],\bar{b})]}}{[a,b,(\bar{a},\bar{b})]}\qquad\text{is dual to}\qquad \mathsf{ai}\uparrow\,\frac{\mathsf{ai}\uparrow\,\frac{\mathsf{s}\,\frac{\mathsf{s}\,\frac{(a,b,[\bar{a},\bar{b}])}{(b,[(a,\bar{a}),\bar{b}])}}{[(a,\bar{a}),(b,\bar{b})]}}{(b,\bar{b})}}{\mathsf{f}}\quad .$$

While atomic rules are good e.g. from the point of view of mechanized proof-search, they are cumbersome for a user of the system. Of course, it should be possible to contract and weaken on arbitrarily large formulas, just as it should

$$\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{i}\!\uparrow \frac{S(R,\bar{R})}{S\{\mathsf{f}\}}$$

$$\mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{w}\!\uparrow \frac{S\{R\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}\!\uparrow \frac{S\{R\}}{S(R,R)}$$

Fig. 3. General identity, weakening, contraction and their duals

be possible to introduce arbitrarily large lemmas through the cut rule. Figure 3 shows the general, i.e. non-atomic, versions of the atomic rules in SKS. The following theorem shows that they can be used.

Theorem 3.2. General identity, weakening, contraction and their duals, i.e. the rules $\{\mathsf{i}\!\downarrow, \mathsf{i}\!\uparrow, \mathsf{w}\!\downarrow, \mathsf{w}\!\uparrow, \mathsf{c}\!\downarrow, \mathsf{c}\!\uparrow\}$ are strongly admissible for system SKS. In particular, the rules $\mathsf{i}\!\downarrow$, $\mathsf{w}\!\downarrow$ and $\mathsf{c}\!\downarrow$ are strongly admissible for $\{\mathsf{ai}\!\downarrow, \mathsf{s}\}$, $\{\mathsf{aw}\!\downarrow, \mathsf{ac}\!\uparrow\}$ and $\{\mathsf{ac}\!\downarrow, \mathsf{m}\}$, respectively. Dually, the rules $\mathsf{i}\!\uparrow$, $\mathsf{w}\!\uparrow$ $\mathsf{c}\!\uparrow$ are strongly admissible for $\{\mathsf{ai}\!\uparrow, \mathsf{s}\}$, $\{\mathsf{aw}\!\uparrow, \mathsf{ac}\!\downarrow\}$ and $\{\mathsf{ac}\!\uparrow, \mathsf{m}\}$, respectively.

Proof. We will show strong admissibility of the rules $\{\mathsf{i}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow\}$ for the respective subsystems of SKS. The proof of strong admissibility of their co-rules is dual.

Given an instance of one of the following rules:

$$\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \quad , \qquad \mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}} \quad , \qquad \mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \quad ,$$

construct a new derivation by structural induction on R:

1. R is a literal: Then the instance of the general rule is also an instance of its atomic form.
2. $R = [P,Q]$, where $P \neq \mathsf{f} \neq Q$: Note that $[\mathsf{f},\mathsf{f}] = \mathsf{f}$. Apply the induction hypothesis respectively on

$$\mathsf{s}\,\frac{\mathsf{s}\,\dfrac{\mathsf{i}\!\downarrow \dfrac{\mathsf{i}\!\downarrow \dfrac{S\{\mathsf{t}\}}{S[Q,\bar{Q}]}}{S([P,\bar{P}],[Q,\bar{Q}])}}{S[Q,([P,\bar{P}],\bar{Q})]}}{S[P,Q,(\bar{P},\bar{Q})]} \quad , \qquad \mathsf{w}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{S\{\mathsf{f}\}}{S[\mathsf{f},Q]}}{S[P,Q]} \quad , \qquad \mathsf{c}\!\downarrow \frac{\mathsf{c}\!\downarrow \dfrac{S[P,P,Q,Q]}{S[P,P,Q]}}{S[P,Q]} \quad .$$

3. $R = (P, Q)$, where $P \neq \mathsf{t} \neq Q$: Apply the induction hypothesis respectively on

$$\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{S\{\mathsf{t}\}}{S[Q,\bar{Q}]}}{S([P,\bar{P}],[Q,\bar{Q}])}}{S[\bar{Q},([P,\bar{P}],Q)]}}{S[\bar{P},\bar{Q},(P,Q)]} \quad , \qquad \mathsf{w}{\downarrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{\mathsf{ac}{\uparrow}\,\dfrac{S\{\mathsf{f}\}}{S(\mathsf{f},\mathsf{f})}}{S(\mathsf{f},Q)}}{S(P,Q)} \quad , \qquad \mathsf{c}{\downarrow}\,\dfrac{\mathsf{c}{\downarrow}\,\dfrac{\mathsf{m}\,\dfrac{S[(P,Q),(P,Q)]}{S([P,P],[Q,Q])}}{S([P,P],Q)}}{S(P,Q)} \quad .$$

□

Example 3.3. Here are two proofs, one using the general rules, the other one in SKS, i.e. without using the general rules:

$$\mathsf{w}{\downarrow}\,\dfrac{\mathsf{c}{\uparrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{[[\bar{a},\bar{b}],(a,b)]}}{[([\bar{a},\bar{b}],[\bar{a},\bar{b}]),(a,b)]}}{[([\bar{a},\bar{b},c,d],[\bar{a},\bar{b}]),(a,b)]} \qquad \text{and} \qquad \mathsf{aw}{\downarrow}\,\dfrac{\mathsf{aw}{\downarrow}\,\dfrac{\mathsf{m}\,\dfrac{\mathsf{ac}{\uparrow}\,\dfrac{\mathsf{ac}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{[\bar{b},b]}}{[\bar{b},([\bar{a},a],b)]}}{[\bar{a},\bar{b},(a,b)]}}{[\bar{a},(\bar{b},\bar{b}),(a,b)]}}{[(\bar{a},\bar{a}),(\bar{b},\bar{b}),(a,b)]}}{[([\bar{a},\bar{b}],[\bar{a},\bar{b}]),(a,b)]}}{[([\bar{a},\bar{b},c],[\bar{a},\bar{b}]),(a,b)]}}{[([\bar{a},\bar{b},c,d],[\bar{a},\bar{b}]),(a,b)]} \quad .$$

4 Equivalence to Classical Logic

In this section we will see translations between system SKS and system GS1p, a Gentzen-Schütte formulation of classical logic [8]. Derivations in GS1p are translated to derivations in SKS (without introducing cuts), and proofs in SKS are translated to proofs in GS1p (possibly introducing cuts). Cut elimination for SKS is a consequence of these translations and cut elimination in GS1p.

System GS1p is shown in Figure 4. Its formulae are denoted by A and B. They contain negation only on atoms. Sequents are denoted by Σ or by $\vdash A_1, \dots, A_h$, where $h \geq 0$. Multisets of formulae are denoted by Φ and Ψ. Derivations are denoted by Δ or $\begin{array}{c}\Sigma_1 \cdots \Sigma_h \\ \triangledown\,\Delta \\ \Sigma\end{array}$, where $h \geq 0$, the sequents $\Sigma_1, \dots, \Sigma_h$ are the premises and Σ is the conclusion. Proofs are denoted by Π.

$$
\mathsf{Ax}\,\frac{}{\vdash A, \bar{A}} \qquad \mathsf{Cut}\,\frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi}
$$

$$
\mathsf{R}\vee_{\mathsf{L}}\,\frac{\vdash \Phi, A}{\vdash \Phi, A \vee B} \qquad \mathsf{R}\vee_{\mathsf{R}}\,\frac{\vdash \Phi, B}{\vdash \Phi, A \vee B} \qquad \mathsf{R}\wedge\,\frac{\vdash \Phi, A \quad \vdash \Phi, B}{\vdash \Phi, A \wedge B}
$$

$$
\mathsf{RC}\,\frac{\vdash \Phi, A, A}{\vdash \Phi, A} \qquad \mathsf{RW}\,\frac{\vdash \Phi}{\vdash \Phi, A}
$$

Fig. 4. GS1p: classical logic in Gentzen-Schütte form

Definition 4.1. The functions $\underline{\ .\ }_{\mathsf{s}}$ and $\underline{\ .\ }_{\mathsf{K}}$ given below transform formulae in GS1p into structures and vice versa:

$$
\begin{array}{rcl}
\underline{a}_{\mathsf{s}} & = & a, \\
\underline{A \vee B}_{\mathsf{s}} & = & [\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}], \\
\underline{A \wedge B}_{\mathsf{s}} & = & (\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}})
\end{array}
\qquad
\begin{array}{rcl}
\underline{a}_{\mathsf{K}} & = & a \\
\underline{\mathsf{t}}_{\mathsf{K}} & = & a^{\bullet} \vee \bar{a}^{\bullet} \\
\underline{\mathsf{f}}_{\mathsf{K}} & = & a^{\bullet} \wedge \bar{a}^{\bullet} \\
\underline{[R,T]}_{\mathsf{K}} & = & \underline{R}_{\mathsf{K}} \vee \underline{T}_{\mathsf{K}}, \\
\underline{(R,T)}_{\mathsf{K}} & = & \underline{R}_{\mathsf{K}} \wedge \underline{T}_{\mathsf{K}},
\end{array}
$$

where $a^{\bullet}$ denotes a fixed arbitrarily chosen atom. The domain of $\underline{\ .\ }_{\mathsf{s}}$ is extended to sequents by $\underline{\vdash}_{\mathsf{s}} = \mathsf{f}$ and $\underline{\vdash A_1, \ldots, A_h}_{\mathsf{s}} = [\underline{A_1}_{\mathsf{s}}, \ldots, \underline{A_h}_{\mathsf{s}}]$, where $h > 0$.

Theorem 4.2. For every derivation $\begin{array}{c}\Sigma_1 \ \cdots \ \Sigma_h \\ \triangledown\,\Delta \\ \Sigma\end{array}$ in GS1p there exists a derivation in SKS $\begin{array}{c}(\underline{\Sigma_1}_{\mathsf{s}}, \ldots, \underline{\Sigma_h}_{\mathsf{s}}) \\ \Vert\,\mathsf{SKS} \\ \underline{\Sigma}_{\mathsf{s}}\end{array}$.

Proof. By structural induction on Δ.

Base Cases

If $\Delta = \Sigma$, take $\underline{\Sigma}_{\mathsf{s}}$, otherwise, if $\Delta = \mathsf{Ax}\,\frac{}{\vdash A, \bar{A}}$ then take $\mathsf{i}{\downarrow}\,\frac{\mathsf{t}}{[\underline{A}_{\mathsf{s}}, \underline{\bar{A}}_{\mathsf{s}}]}$.

Inductive Cases

We show the case where $\Delta = \begin{array}{c} \begin{array}{cc} \Sigma'_1 \ \cdots \ \Sigma'_k & \Sigma''_1 \ \cdots \ \Sigma''_l \\ \triangledown & \triangledown \end{array} \\ \mathsf{R}\wedge\,\dfrac{\vdash \Phi, A \qquad\qquad \vdash \Phi, B}{\vdash \Phi, A \wedge B} \end{array}$. The translations for other cases can be done in a similar way. By inductive hypothesis, we have the following derivations:

$$\begin{array}{c}(\underline{\Sigma'_1}_{\mathsf{s}},\ldots,\underline{\Sigma'_k}_{\mathsf{s}})\\ \Delta_1\big\Vert\mathsf{SKS}\\ [\underline{\Phi}_{\mathsf{s}},\underline{A}_{\mathsf{s}}]\end{array}\quad\text{and}\quad\begin{array}{c}(\underline{\Sigma''_1}_{\mathsf{s}},\ldots,\underline{\Sigma''_l}_{\mathsf{s}})\\ \Delta_2\big\Vert\mathsf{SKS}\\ [\underline{\Phi}_{\mathsf{s}},\underline{B}_{\mathsf{s}}]\end{array}\quad\Rightarrow\quad\begin{array}{c}(\underline{\Sigma'_1}_{\mathsf{s}},\ldots,\underline{\Sigma'_k}_{\mathsf{s}},\underline{\Sigma''_1}_{\mathsf{s}},\ldots,\underline{\Sigma''_l}_{\mathsf{s}})\\ \Delta_1;\Delta_2\big\Vert\mathsf{SKS}\\ \mathsf{c}\!\downarrow\dfrac{\mathsf{s};\mathsf{s}\,\dfrac{([\underline{\Phi}_{\mathsf{s}},\underline{A}_{\mathsf{s}}],[\underline{\Phi}_{\mathsf{s}},\underline{B}_{\mathsf{s}}])}{[\underline{\Phi}_{\mathsf{s}},\underline{\Phi}_{\mathsf{s}},(\underline{A}_{\mathsf{s}},\underline{B}_{\mathsf{s}})]}}{[\underline{\Phi}_{\mathsf{s}},(\underline{A}_{\mathsf{s}},\underline{B}_{\mathsf{s}})]}\end{array}\ .$$

□

Corollary 4.3. If $\vdash A$ is provable in $\mathtt{GS1p}$ then $\underline{\vdash A}_{\mathsf{s}}$ is provable in SKS.

Theorem 4.4. If P is provable in SKS then $\vdash \underline{P}_{\mathsf{K}}$ is provable in $\mathtt{GS1p}$.

Proof. Let $P = S\{R\}$ and $\begin{array}{c}\Pi\big\Vert\mathsf{SKS}\\ \rho\,\dfrac{S\{T\}}{S\{R\}}\end{array}$ be its proof in SKS. The proof of this theorem is based on a known property of $\mathtt{GS1p}$, that is, if $\vdash \underline{R}_{\mathsf{K}},\overline{\underline{T}_{\mathsf{K}}}$ is provable then so is $\vdash \underline{S\{R\}}_{\mathsf{K}},\overline{\underline{S\{T\}}_{\mathsf{K}}}$.

Base Cases

If $\Pi = \mathsf{t}$ then take $\mathsf{RC}\,\dfrac{\mathsf{R\vee_L};\mathsf{R\vee_R}\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash a^\bullet,\bar{a}^\bullet}}{\vdash a^\bullet\vee\bar{a}^\bullet,a^\bullet\vee\bar{a}^\bullet}}{a^\bullet\vee\bar{a}^\bullet}$, otherwise, if $\Pi = \mathsf{ai}\!\downarrow\dfrac{\mathsf{t}}{[a,\bar{a}]}$ then take the same derivation, but with $a^\bullet$ replaced by a.

Inductive Cases

We assume that $\underline{S\{T\}}_{\mathsf{K}}$ is provable in $\mathtt{GS1p}$. By using the cut rule, we get $\mathsf{Cut}\,\dfrac{\vdash \underline{S\{T\}}_{\mathsf{K}}\quad \vdash \underline{S\{R\}}_{\mathsf{K}},\overline{\underline{S\{T\}}_{\mathsf{K}}}}{\vdash \underline{S\{R\}}_{\mathsf{K}}}$. It is enough to show that $\vdash \underline{R}_{\mathsf{K}},\underline{\bar{T}}_{\mathsf{K}}$ is provable. We show the case for $\rho = \mathsf{s}\,\dfrac{S([U,V],W)}{S[(U,W),V]}$. The property holds for the rest of the rules of SKS as well, as can easily be verified.

$$\mathsf{RC}^2\,\dfrac{\mathsf{R\vee_L}^2;\mathsf{R\vee_R}^2\,\dfrac{\mathsf{R\wedge}\,\dfrac{\mathsf{R\wedge}\,\dfrac{\mathsf{RW}^2\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash \underline{U}_{\mathsf{K}},\overline{\underline{U}_{\mathsf{K}}}}}{\vdash \underline{U}_{\mathsf{K}},\overline{\underline{U}_{\mathsf{K}}},\underline{V}_{\mathsf{K}},\overline{\underline{W}_{\mathsf{K}}}}\quad \mathsf{RW}^2\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash \underline{V}_{\mathsf{K}},\overline{\underline{V}_{\mathsf{K}}}}}{\vdash \underline{V}_{\mathsf{K}},\overline{\underline{V}_{\mathsf{K}}},\underline{U}_{\mathsf{K}},\overline{\underline{W}_{\mathsf{K}}}}}{\vdash \underline{U}_{\mathsf{K}},\underline{V}_{\mathsf{K}},\overline{\underline{U}_{\mathsf{K}}}\wedge\overline{\underline{V}_{\mathsf{K}}},\overline{\underline{W}_{\mathsf{K}}}}\quad \mathsf{RW}^2\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash \underline{W}_{\mathsf{K}},\overline{\underline{W}_{\mathsf{K}}}}}{\vdash \underline{W}_{\mathsf{K}},\overline{\underline{W}_{\mathsf{K}}},\underline{V}_{\mathsf{K}},\overline{\underline{U}_{\mathsf{K}}}\wedge\overline{\underline{V}_{\mathsf{K}}}}}{\vdash \underline{U}_{\mathsf{K}}\wedge\underline{W}_{\mathsf{K}},\underline{V}_{\mathsf{K}},\overline{\underline{U}_{\mathsf{K}}}\wedge\overline{\underline{V}_{\mathsf{K}}},\overline{\underline{W}_{\mathsf{K}}}}}{\vdash (\underline{U}_{\mathsf{K}}\wedge\underline{W}_{\mathsf{K}})\vee\underline{V}_{\mathsf{K}},(\overline{\underline{U}_{\mathsf{K}}}\wedge\overline{\underline{V}_{\mathsf{K}}})\vee\overline{\underline{W}_{\mathsf{K}}},(\underline{U}_{\mathsf{K}}\wedge\underline{W}_{\mathsf{K}})\vee\underline{V}_{\mathsf{K}},(\overline{\underline{U}_{\mathsf{K}}}\wedge\overline{\underline{V}_{\mathsf{K}}})\vee\overline{\underline{W}_{\mathsf{K}}}}}{\vdash (\underline{U}_{\mathsf{K}}\wedge\underline{W}_{\mathsf{K}})\vee\underline{V}_{\mathsf{K}},(\overline{\underline{U}_{\mathsf{K}}}\wedge\overline{\underline{V}_{\mathsf{K}}})\vee\overline{\underline{W}_{\mathsf{K}}}}\ .$$

The rule ρ^n denotes n applications of ρ. □

Cut elimination for SKS can be obtained by using the above translations: Given a proof Π in SKS, we can transform it to a proof Π' in GS1p and eliminate all the cuts there. The resulting cut-free proof in GS1p can then be translated back to a proof Π'' in SKS. The complete case analysis of the proof of Theorem 4.2 shows that this transformation does not produce new cuts, and hence Π'' is a cut-free proof in SKS.

5 Cut Elimination and Decomposition

There is a very natural way of proving cut elimination for system SKS by using semantics, using the idea employed in [8] for the system G3. The proof actually gives us something more than just cut elimination, it eliminates all up-rules and also yields a decomposition of proofs into separate phases.

Theorem 5.1 (Cut Elimination, semantically).

$$\text{For every proof } \begin{array}{c} \overline{\Vert}\,\mathsf{SKS} \\ S \end{array} \text{ there is a proof } \begin{array}{c} \overline{\Vert}\,\{\mathsf{ai}\!\downarrow\} \\ S'' \\ \Vert\,\{\mathsf{aw}\!\downarrow\} \\ S' \\ \Vert\,\{\mathsf{s},\mathsf{ac}\!\downarrow,\mathsf{m}\} \\ S \end{array} \quad .$$

Proof. Consider the rule *distribute*

$$\mathsf{d}\!\downarrow\,\frac{S([R,T],[R,U])}{S[R,(T,U)]},$$

which can be realized by a contraction and two switches:

$$\mathsf{c}\!\downarrow\,\frac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{S([R,T],[R,U])}{S[R,([R,T],U)]}}{S[R,R,(T,U)]}}{S[R,(T,U)]}$$

and thus by Theorem 3.2 is strongly admissible for $\{\mathsf{s},\mathsf{ac}\!\downarrow,\mathsf{m}\}$. Build a derivation $\begin{array}{c} S' \\ \Vert\,\{\mathsf{d}\!\downarrow\} \\ S \end{array}$, by going upwards from S applying $\mathsf{d}\!\downarrow$ as many times as possible. Then S' will be in conjunctive normal form, i.e.

$$S' = ([a_{11},a_{12},\ldots],[a_{21},a_{22},\ldots],\ldots,[a_{n1},a_{n2},\ldots]) \quad .$$

S is valid because there is a proof of it. The rule $\mathsf{d}\!\downarrow$ is invertible, so S' is also valid. A conjunction is valid only if all its immediate substructures are valid. Those are disjunctions of atoms. A disjunction of atoms is valid only if it contains an atom a together with its negation $\bar{a}$. Thus, more specifically, S' is of the form

$$S' = ([b_1, \bar{b}_1, a_{11}, a_{12}, \ldots], [b_2, \bar{b}_2, a_{21}, a_{22}, \ldots], \ldots, [b_n, \bar{b}_n, a_{n1}, a_{n2}, \ldots]) \quad .$$

Let $S'' = ([b_1, \bar{b}_1], [b_2, \bar{b}_2], \ldots, [b_n, \bar{b}_n]) \quad .$

Obviously, there is a derivation $\begin{array}{c} S'' \\ \Vert \{\mathsf{aw}\!\downarrow\} \\ S' \end{array}$ and a proof $\begin{array}{c} \overline{\Vert} \{\mathsf{ai}\!\downarrow\} \\ S'' \end{array}$. □

Let us call system KS the rules shown in Fig. 5. We know that for proof-search in SKS system KS is sufficient:

Corollary 5.2. For every proof $\begin{array}{c} \overline{\Vert} \mathsf{SKS} \\ S \end{array}$ there is a proof $\begin{array}{c} \overline{\Vert} \mathsf{KS} \\ S \end{array}$.

As a result of cut elimination, sequent systems fulfill the subformula property. Our case is different, because our rules do not split the derivation according to the main connective of the active formula. However, system KS satisfies the main consequence of the subformula property: no new atoms have to be introduced in proof-search, i.e. the branching of the search tree is finite.

$$\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S[a, \bar{a}]} \qquad \mathsf{aw}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{ac}\!\downarrow \frac{S[a, a]}{S\{a\}}$$

$$\mathsf{s} \frac{S([R, T], U)}{S[(R, U), T]} \qquad \mathsf{m} \frac{S[(R, T), (U, V)]}{S([R, U], [T, V])}$$

Fig. 5. System KS

Given that in system SKS the identity is a rule, not an axiom as in the sequent calculus, a natural question to ask is whether the applications of the identity rule can be restricted to the top of a derivation. For proofs, this question is already answered positively by Theorem 5.1. It turns out that it is also true for derivations. Because of the duality between $\mathsf{ai}\!\downarrow$ and $\mathsf{ai}\!\uparrow$ we can also push the cuts to the bottom of a derivation. While this can be obtained in the sequent calculus (using cut elimination), it can not be done with a simple permutation argument.

We first reduce atomic identity and cut to *shallow* atomic identity and cut, the following rules:

$$\mathsf{ai_s}\!\downarrow \frac{S}{(S, [a, \bar{a}])} \qquad \text{and} \qquad \mathsf{ai_s}\!\uparrow \frac{[S, (a, \bar{a})]}{S} \quad .$$

Lemma 5.3. The rule $\mathsf{ai}\!\downarrow$ is strongly admissible for $\{\mathsf{ai_s}\!\downarrow, \mathsf{s}\}$. Dually, the rule $\mathsf{ai}\!\uparrow$ is strongly admissible for $\{\mathsf{ai_s}\!\uparrow, \mathsf{s}\}$.

Proof. By an easy structural induction on the context $S\{\ \}$. Details are in [1]. □

Theorem 5.4 (Decomposition: separation of identity and cut).

$$\text{For every derivation } \begin{array}{c} T \\ \| \mathsf{SKS} \\ R \end{array} \text{ there is a derivation } \begin{array}{c} T \\ \| \{\mathsf{ai}\downarrow\} \\ V \\ \| \mathsf{SKS} \setminus \{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow\} \\ U \\ \| \{\mathsf{ai}\uparrow\} \\ R \end{array} .$$

Proof. By Lemma 5.3 we can reduce atomic identities to shallow atomic identities and the same for the cuts. It is easy to check that the rule $\mathsf{ai_s}\!\downarrow$ permutes over every rule in SKS and the rule $\mathsf{ai_s}\!\uparrow$ permutes under every rule in SKS. Instances of $\mathsf{ai_s}\!\downarrow$ and $\mathsf{ai_s}\!\uparrow$ are instances of $\mathsf{ai}\!\downarrow$ and $\mathsf{ai}\!\uparrow$, respectively. □

Contraction allows the repeated use of a statement in a proof by allowing to copy it at will. It should be possible to copy everything needed in the beginning, and then go on with the proof without ever having to copy again. This intuition is made precise by the following theorem and holds for system SKS. We do not know of such a result for the sequent calculus. There are sequent systems for classical propositional logic that do not have an explicit contraction rule, however, they treat the context additively, so contraction is "built-in" and used throughout the proof.

Theorem 5.5 (Decomposition: separation of atomic contraction).

$$\text{For every derivation } \begin{array}{c} T \\ \| \mathsf{SKS} \\ R \end{array} \text{ there is a derivation } \begin{array}{c} T \\ \| \{\mathsf{ac}\uparrow\} \\ V \\ \| \mathsf{SKS} \setminus \{\mathsf{ac}\downarrow, \mathsf{ac}\uparrow\} \\ U \\ \| \{\mathsf{ac}\downarrow\} \\ R \end{array} .$$

Proof. The obstacles to permuting up the instances of $\mathsf{ac}\!\uparrow$ and down those of $\mathsf{ac}\!\downarrow$ are identity and cut, respectively. The solution is to turn the derivation into a proof, eliminate the cuts, turn the proof into a derivation again (using one cut), and then permuting up or down the contractions. The proof can be found in [1]. □

6 Conclusions and Open Problems

We have presented SKS, a system of inference rules for classical logic in the calculus of structures. Its main novelty is that all rules are local and their computational cost can thus be bounded. To achieve this, the greater expressivity of the calculus of structures wrt. the sequent calculus was used, in particular its ability of making deep inferences. We proved cut elimination for system SKS which makes it suitable for proof-search. Actually, a subset KS of inference rules is already complete. We have also shown properties of our system that seem not to hold for any sequent presentation of classical logic, that is, strong admissibility of cut, weakening and contraction for their atomic forms and the decomposition theorems for derivations.

The main open problem is a more powerful decomposition theorem. To that end, let us call *core* those rules in the system that are necessary for decomposing the general cut into atomic cuts. In SKS, the core consists of one single rule: the switch. Can we separate out, i.e. push above the identities or below the cuts, anything that is not core?

Conjecture 6.1. For every derivation $\begin{array}{c} T \\ \Vert \mathsf{SKS} \\ R \end{array}$ there is a derivation $\begin{array}{c} T \\ \Vert \mathit{non\text{-}core} \\ U_4 \\ \Vert \{\mathsf{ai}{\downarrow}\} \\ U_3 \\ \Vert \mathit{core} \\ U_2 \\ \Vert \{\mathsf{ai}{\uparrow}\} \\ U_1 \\ \Vert \mathit{non\text{-}core} \\ R \end{array}$.

This conjecture has been proved for two other systems in the calculus of structures [6] and this led to cut elimination. In these cut elimination proofs, atomic cuts are seen as instances of a super atomic cut, which is then pushed up all the way through the proof until it hits an identity that makes it disappear. In system SKS, such a super atomic cut cannot be pushed up over the rules $\mathsf{ac}{\downarrow}$ and m. Cut elimination would be much easier to prove syntactically could we rely on Conjecture 6.1. Then all the problematic rules that could stand in the way of the cut are either below all the cuts already or at the top of the proof and thus trivial, since their premise is t. Cut elimination is thus an easy consequence of such a decomposition theorem. Note that the proof of Theorem 5.5 falls short of simplifying a syntactical proof of cut elimination not only because instances of the rule m remain above the cuts, but also because it *uses* cut elimination.

Modularity We have proved cut elimination for system SKS, but we have no syntactic proof inside the calculus of structures, i.e. without detour through the sequent calculus and without resorting to semantics. We are interested in such a proof because it can be *modular*, contrary to cut elimination proofs in the sequent calculus, cf. Girard [4] p.15. This modularity stems from the fact that due to atomicity of the cut, cut elimination in the calculus of structures is not a nested induction taking into account the cut rank; instead it is based on a number of lemmas about permutability of rules wrt. one another (for a rather general notion of permutability). Those lemmas of course are not affected when new rules are added to the system.

Predicative logic We are currently investigating the following extension of system SKS to predicative logic: adding quantifiers to the language in the obvious way, adding the corresponding De Morgan laws and the equation

$$\forall x R = \exists x R = R \qquad \text{if } x \text{ is not free in } R,$$

and adding the rules from Fig. 6. Very roughly, rules $\{\mathsf{u}\!\downarrow, \mathsf{u}\!\uparrow\}$ correspond to the $\mathsf{R}\forall$ rule in GS1 while rules $\{\mathsf{n}\!\downarrow, \mathsf{n}\!\uparrow\}$ correspond to $\mathsf{R}\exists$. The rules $\{\mathsf{ce}\!\downarrow, \mathsf{ce}\!\uparrow, \mathsf{ca}\!\downarrow, \mathsf{ca}\!\uparrow\}$ are just needed to reduce contraction to its atomic form. For proofs, the up-rules $\{\mathsf{n}\!\uparrow, \mathsf{u}\!\uparrow, \mathsf{ce}\!\uparrow, \mathsf{ca}\!\uparrow\}$ are admissible. A nice common feature of all these rules is that their premise implies their conclusion (literally, without any added quantification). This is not true of any sequent calculus presentation known to us because of the $\mathsf{R}\forall$ rule.

We do not claim that this system is local. In the rule $\mathsf{n}\!\downarrow$ a term t of unbounded size is copied into an unbounded number of occurrences of x in R. Maybe unification could be incorporated into the system to deal with this in a local manner, but we have not explored this option. The question is whether this can be done without losing the good properties, cut elimination especially.

$$\mathsf{u}\!\downarrow \frac{S\{\forall x[R,T]\}}{S[\forall x R, \exists x T]} \qquad \mathsf{u}\!\uparrow \frac{S(\exists x R, \forall x T)}{S\{\exists x(R,T)\}} \qquad \mathsf{n}\!\downarrow \frac{S\{R[x \leftarrow t]\}}{S\{\exists x R\}} \qquad \mathsf{n}\!\uparrow \frac{S\{\forall x R\}}{S\{R[x \leftarrow t]\}}$$

$$\mathsf{ce}\!\downarrow \frac{S[\exists x R, \exists x T]}{S\{\exists x[R,T]\}} \qquad \mathsf{ce}\!\uparrow \frac{S\{\forall x(R,T)\}}{S(\forall x R, \forall x T)} \qquad \mathsf{ca}\!\downarrow \frac{S[\forall x R, \forall x T]}{S\{\forall x[R,T]\}} \qquad \mathsf{ca}\!\uparrow \frac{S\{\exists x(R,T)\}}{S(\exists x R, \exists x T)}$$

Fig. 6. Extension to predicative logic

Semantics for derivations Structures are in a one-to-one correspondence with *traces* [5] that are graphs with colored edges satisfying certain simple properties. The atom occurrences of a structure are the nodes of its trace and the colors of the edges are determined by the logical relation between the atom occurrences. In [5] it is shown that the switch rule can be characterized in terms of conditions on traces. Those conditions can be checked locally in the sense that they involve at most four atoms at a time.

The question is whether the rule m can be characterized in the same way. This would be a step towards a distributed system in which proof-search is driven by pairs of complementary atoms, comparable in spirit to the connection method [9]. At present, however, this question is entirely open.

Hopefully, trace semantics can help in understanding derivations. Given the existence of a derivation in a subset of SKS from a known S to an unknown T, what is the relation between (the traces of) S and T? What can be inferred about T, i.e. what graph-theoretic properties on traces are preserved by the inference rules? By classical semantics we know that all of them preserve truth (successful valuations). The problem is that this does not tell us much about T, in particular it tells us nothing about atom occurrences, their number, and their logical relations. A better understanding of this would also help in finding a decomposition theorem as sketched in Conjecture 6.1.

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References

1. Kai Brünnler and Alwen Fernanto Tiu. A local system for classical logic. Technical Report WV-2001-02, Dresden University of Technology, 2001. On the web at: http://www.wv.inf.tu-dresden.de/~kai/LocalClassicalLogic.ps.gz.
2. Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland, Amsterdam, 1969.
3. Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
4. Jean-Yves Girard. *Proof Theory and Logical Complexity, Volume I*, volume 1 of *Studies in Proof Theory*. Bibliopolis, Napoli, 1987. Distributed by Elsevier.
5. Alessio Guglielmi. A calculus of order and interaction. Technical Report WV-99-04, Dresden University of Technology, 1999. Available on the web at http://www.wv.inf.tu-dresden.de/~guglielm/Research/Gug/Gug.pdf.
6. Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. Technical Report WV-01-04, Dresden University of Technology, 2001. Accepted by the Annual Conference of the European Association for Computer Science Logic, CSL'01.
7. Lutz Straßburger. MELL in the calculus of structures. Technical Report WV-2001-03, Dresden University of Technology, 2001. On the web at: http://www.ki.inf.tu-dresden.de/~lutz/els.pdf.
8. Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 1996.
9. W. Bibel. On matrices with connections. *Journal of the Association for Computing Machinery*, 28(4):633–645, 1981.