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Deep Sequent Systems for Modal Logic

Kai Brünnler

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Abstract We see a systematic set of cut-free axiomatisations for all the basic normal modal logics formed by some combination the axioms $\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}$. They employ a form of deep inference but otherwise stay very close to Gentzen's sequent calculus, in particular they enjoy a subformula property in the literal sense. No semantic notions are used inside the proof systems, in particular there is no use of labels. All their rules are invertible and the rules cut, weakening and contraction are admissible. All systems admit a straightforward terminating proof search procedure as well as a syntactic cut elimination procedure.

Keywords nested sequents · modal logic · cut elimination · deep inference

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1 Introduction

Numerous extensions of the sequent calculus have been proposed in order to give cut-free axiomatisations of modal logic. They are divided into two classes: *labelled* formalisms, which incorporate Kripke semantics in the proof system, and *unlabelled* formalisms, which do not. Prominent examples of unlabelled formalisms are the hypersequent calculus [1] and the display calculus [2,20]. These and more can be found in the survey by Wansing [21]. A recent account of labelled sequent systems which also includes more references can be found in Negri [13].

The labelled approach seems to have become more prominent and according to several criteria has been more successful than the unlabelled approaches. It allows to capture a wide class of modal logics and does so systematically. In many important cases it yields systems which are natural and easy to use, which have good structural properties like contraction-admissibility and invertibility of all rules, and which give rise to decision procedures.

Institut für angewandte Mathematik und Informatik
Neubrückstr. 10, CH – 3012 Bern, Switzerland
http://www.iam.unibe.ch/~kai/

Labelled systems are formulated in a hybrid language which not only contains modalities but also variables and an accessibility relation. There are some concerns about incorporating the semantics into the syntax of a proof system in this way. Avron discusses them in [1], for example. However, even without these concerns, it still is an interesting question whether we can formulate certain proof systems for modal logic within the modal language or whether we have to move to the hybrid language. Thus my goal here is to develop proof systems with the same good properties of Negri's labelled systems but to do so within the modal language.

This paper is part of a research effort started by Hein, Stewart and Stouppa [10, 16, 17]. To make the property of "not using labels" a bit more precise we call a proof system *pure* if each sequent has an obvious corresponding formula. Ordinary sequent systems for modal logic are clearly pure: just read the comma on the left as conjunction, the comma on the right as disjunction, and the turnstile as implication. Hypersequents are also pure. For a labelled sequent, on the other hand, it is generally not clear what its corresponding modal formula should be.

Hein, Stewart and Stouppa use the *calculus of structures* to give pure systems for modal logics. This formalism was developed by Guglielmi [8] and was further studied by the author [5,3] and by Straßburger[9,18]. It is based on *deep inference*, which is the ability to apply rules deep inside of a formula. So far the calculus of structures has captured essentially those modal logics which can also be captured using the sequent calculus or hypersequents. In particular that does not include B and $\mathsf{K5}$. It turns out that not all the depth of the calculus of structures is needed for our purpose and so we use proof systems based on *nested sequents* which are intermediate between the calculus of structures and the sequent calculus. Essentially the same notion of sequent has been used before by Bull [6] to give a proof system for a fragment of propositional dynamic logic and by Kashima [11] to give proof systems for certain tense logics. In the current work we will see deep sequent systems for all the normal logics formed from the axioms $\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}$, so the modal logics shown in Figure 1. In particular this includes B and $\mathsf{K5}$. All proof systems can be easily embedded into corresponding systems in the calculus of structures, as we will see, so this answers questions from Stewart and Stouppa [16].

The plan of the paper is as follows: after some preliminaries I present deep sequent systems and prove invertibility of rules and admissibility of contraction. Then I show that they are sound and complete for the respective Kripke semantics. The completeness proof constructs a countermodel from the failure of a terminating proof search procedure. After that we see the syntactic cut elimination procedure in the course of which we find some admissible structural rules, which are interesting in their own right. These modal structural rules might lead to modular systems, as we see in the next section. Some discussion of related formalisms ends this paper. This paper is an extended and corrected version of [4]. In addition to [4] I also treat seriality, give a syntactic cut elimination procedure, give systems based on the modal structural rules and make explicit the connection to the calculus of structures.

2 The Sequent Systems

Formulas. Propositions p and their negations $\bar{p}$ are *atoms*, with $\bar{\bar{p}}$ defined to be p. *Formulas*, denoted by A, B, C, D are given by the grammar

$$A ::= p \mid \bar{p} \mid (A \vee A) \mid (A \wedge A) \mid \Diamond A \mid \Box A \quad .$$

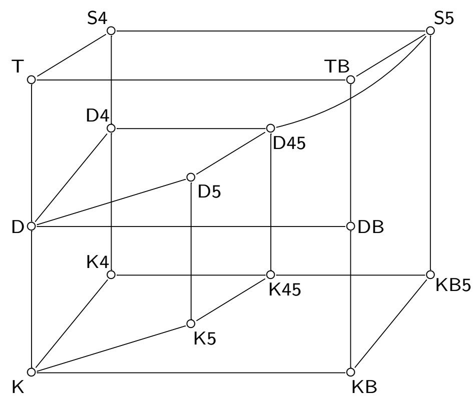


Fig. 1 The modal "cube" [7]

k:	(no condition)	$\top$	$\Box(A \vee B) \supset (\Box A \vee \Diamond B)$
d:	serial	$\forall s \exists t.\ s \to t$	$\Box A \supset \Diamond A$
t:	reflexive	$\forall s.\ s \to s$	$A \supset \Diamond A$
b:	symmetric	$\forall st.\ s \to t \supset t \to s$	$A \supset \Box \Diamond A$
4:	transitive	$\forall stu.\ s \to t \wedge t \to u \supset s \to u$	$\Box A \supset \Box\Box A$
5:	euclidean	$\forall stu.\ s \to t \wedge s \to u \supset t \to u$	$\Diamond A \supset \Box \Diamond A$

Fig. 2 Frame conditions and modal axioms

Given a formula A, its *negation* $\bar{A}$ is defined as usual using the De Morgan laws, $A \supset B$ is defined as $\bar{A} \vee B$ and $\bot$ and $\top$ are defined as $p \wedge \bar{p}$ and $p \vee \bar{p}$, respectively, for some proposition p.

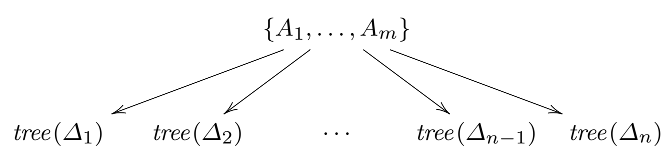
Nested sequents. A *nested sequent* is a finite multiset of formulas and boxed sequents. A *boxed sequent* is an expression $[\Gamma]$ where Γ is a deep sequent. In the following a *sequent* is a nested sequent. Sequents are denoted by $\Gamma, \Delta, \Lambda, \Pi, \Sigma$. As usual, sequents are written without any curly braces and the comma in the expression Γ, Δ is multiset union. A sequent is always of the form

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n] \quad .$$

The *corresponding formula* of the above sequent is $\bot$ if $m = n = 0$ and otherwise

$$A_1 \vee \cdots \vee A_m \vee \Box(D_1) \vee \cdots \vee \Box(D_n) \quad ,$$

where $D_1 \ldots D_n$ are the corresponding formulas of the sequents $\Delta_1 \ldots \Delta_n$. Often we do not distinguish between a sequent and its corresponding formula, for example a model of a sequent is a model of its corresponding formula. A sequent Γ has a *corresponding tree*, denoted *tree*(Γ), whose nodes are marked with multisets of formulas. The corresponding tree of the above sequent is



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Often we do not distinguish between a sequent and its corresponding tree, for example the *root* of a sequent is the root of its corresponding tree.

Sequent contexts. Informally, a context is a sequent with holes. We will mostly encounter sequents with just one hole. A *unary context* is a sequent with exactly one occurrence of the symbol $\{\ \}$, the *hole*, which does not occur inside formulas. Such contexts are denoted by $\Gamma\{\ \}$, $\Delta\{\ \}$, and so on. The hole is also called the *empty context*. The sequent $\Gamma\{\Delta\}$ is obtained by replacing $\{\ \}$ inside $\Gamma\{\ \}$ by Δ. For example, if $\Gamma\{\ \} = A, [[B], \{\ \}]$ and $\Delta = C, [D]$ then

$$\Gamma\{\Delta\} = A, [[B], C, [D]] \quad .$$

The *depth* of a unary context $\Gamma\{\ \}$, denoted $depth(\Gamma\{\ \})$ is defined as follows

$$\begin{aligned} &depth(\Gamma, \{\ \}) = 0 \\ &depth(\Gamma, [\Delta\{\ \}]) = depth(\Delta\{\ \}) + 1. \end{aligned}$$

More generally, a *context* is a sequent with $n \geq 0$ occurrences of $\{\ \}$, which do not occur inside formulas, and which are linearly ordered. The number n is the *arity* of the context. A context is either denoted by $\mathcal{C}$ or by

$$\Gamma \underbrace{\{\ \} \dots \{\ \}}_{n-\text{times}} \quad ,$$

if it is of arity n. Given n contexts $\mathcal{C}_1, \dots, \mathcal{C}_n$ the context

$$\Gamma\{\mathcal{C}_1\} \dots \{\mathcal{C}_n\}$$

is obtained by replacing the i-th hole in $\Gamma\{\ \} \dots \{\ \}$ by $\mathcal{C}_i$ and the i-th element in the linear order by the linear order given by $\mathcal{C}_i$ and doing so simultaneously for all $1 \leq i \leq n$. Clearly the arity of that context is the sum of the arities of all the $\mathcal{C}_i$'s. If a $\mathcal{C}_i$ is the empty context then it is not shown. For example, if $\Gamma\{\ \}\{\ \} = A, [[B], \{\ \}], \{\ \}$ and $\Delta\{\ \} = C, [\{\ \}]$ then

$$\Gamma\{\Delta\{\ \}\}\{\ \} = A, [[B], C, [\{\ \}]], \{\ \} \quad ,$$

where in all contexts the holes are ordered from left to right as shown.

The sequent systems. Figure 3 shows the set of rules from which we form our deductive systems. *System* K is the set of rules $\{\wedge, \vee, \Box, \mathsf{k}\}$. We will look at extensions of System K with sets of rules $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. Each name in X not only designates a rule name, but also a frame condition and a modal Hilbert-style axiom as shown in Figure 2.

In the following instance of an inference rule ρ

$$\rho \frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Delta}$$

we call $\Gamma_1 \dots \Gamma_n$ its *premises* and Δ its *conclusion*. A *system*, denoted by $\mathcal{S}$, is a set of inference rules. A *derivation* in a system $\mathcal{S}$ is a upward-growing finite tree whose nodes are labelled with sequents and which is built according to the inference rules from $\mathcal{S}$. The sequent at the root is the *conclusion* and the sequents at the leaves are the *premises* of the derivation. A *proof* of a sequent Γ in a system is a derivation in this system with conclusion Γ and where all premises are instances of the axiom $\Gamma\{p, \bar{p}\}$.

$$\Gamma\{p,\bar{p}\} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}}$$

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \mathsf{k}\,\frac{\Gamma\{\Diamond A,[\Delta,A]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

$$\mathsf{d}\,\frac{\Gamma\{\Diamond A,[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}\,\frac{\Gamma\{\Diamond A,A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}\,\frac{\Gamma\{[\Delta,\Diamond A],A\}}{\Gamma\{[\Delta,\Diamond A]\}}$$

$$\mathsf{4}\,\frac{\Gamma\{\Diamond A,[\Delta,\Diamond A]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \mathsf{5}\,\frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \ \mathit{depth}(\Gamma\{\ \}\{\emptyset\})>0$$

Fig. 3 System K+{$\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}$}

$$\mathsf{nec}\,\frac{\Gamma}{[\Gamma]} \qquad \mathsf{wk}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

Fig. 4 Necessitation, weakening and contraction

Derivations are denoted by δ and proofs by π. The depth of a derivation δ is denoted by $|\delta|$. Note that the depth of a derivation, which is a tree, has nothing to do with the depth of the sequents in it, which are also trees. We write $\mathcal{S} \vdash \Gamma$ if there is a proof of Γ in system $\mathcal{S}$. An inference rule ρ is *(depth-preserving) admissible* for a system $\mathcal{S}$ if for each proof in $\mathcal{S} \cup \{\rho\}$ there is a proof of in $\mathcal{S}$ with the same conclusion (and with at most the same depth). For each rule ρ there is its *inverse*, denoted by $\bar{\rho}$, which is obtained by exchanging premise and conclusion. The $\bar{\wedge}$-rule allows both $\Gamma\{A\}$ and $\Gamma\{B\}$ as conclusions of $\Gamma\{A \wedge B\}$. An inference rule ρ is *(depth-preserving) invertible* for a system $\mathcal{S}$ if $\bar{\rho}$ is (depth-preserving) admissible for $\mathcal{S}$.

The rules shown in Figure 4 turn out to be admissible.

Lemma 1 (Admissibility of structural rules, Invertibility) *For each system* $\mathsf{K}+\mathsf{X}$ *with* $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *the following hold:*
(i) The rules necessitation, weakening and contraction are depth-preserving admissible.
(ii) All its rules are depth-preserving invertible.

Proof The admissibility of necessitation and weakening follows from a routine induction on the depth of the proof. The same works for the invertibility of the $\wedge, \vee$ and $\Box$-rules in (ii). The inverses of all other rules are just weakenings. For the admissibility of contraction we also proceed by induction on the depth of the proof tree, using depth-preserving invertibility of the rules. The cases are easy for the propositional rules and for the $\Box, \mathsf{d}, \mathsf{t}$-rules. For the k rule we consider the formula $\Diamond A$ from its conclusion $\Gamma\{\Diamond A, [\Delta]\}$ and its position inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have the cases 1) $\Diamond A$ is inside Σ or 2) $\Diamond A$ is inside $\Lambda\{\ \}$. We have three subcases for case 1: 1.1) $[\Delta]$ inside $\Lambda\{\ \}$, 1.2) $[\Delta]$ inside Σ, 1.3) Σ, Σ inside $[\Delta]$. There are two subcases of case 2: 2.1) $[\Delta]$ inside $\Lambda\{\ \}$ and 2.2) $[\Delta]$ inside Σ. All cases are either simpler than or similar to case 1.2, which is as follows:

$$\mathsf{ctr}\,\dfrac{\mathsf{k}\,\dfrac{\Lambda'\{\Diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta]\}}{\Lambda'\{\Diamond A, \Sigma', [\Delta], \Sigma', [\Delta]\}}}{\Lambda'\{\Diamond A, \Sigma', [\Delta]\}} \quad \rightsquigarrow \quad \mathsf{k}\,\dfrac{\mathsf{ctr}\,\dfrac{\bar{\mathsf{k}}\,\dfrac{\Lambda'\{\Diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta]\}}{\Lambda'\{\Diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta, A]\}}}{\Lambda'\{\Diamond A, \Sigma', [\Delta, A]\}}}{\Lambda'\{\Diamond A, \Sigma', [\Delta]\}} \quad ,$$

where the instance of $\bar{\mathsf{k}}$ in the proof on the right is removed because it is depth-preserving admissible and the instance of contraction is removed by the induction hypothesis. The case for the 4-rule works the same way.

For the b-rule we make a case analysis based on the position of $[\Delta, \Diamond A]$ from its conclusion $\Gamma\{[\Delta, \Diamond A]\}$ inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have three cases: 1) $[\Delta, \Diamond A]$ inside $\Lambda\{\ \}$, 2) $[\Delta, \Diamond A]$ in Σ and 3) Σ, Σ inside $[\Delta, \Diamond A]$. Case 3 has two subcases: either $\Diamond A \in \Sigma$ or not. All cases are trivial except for case 2 where invertibility of the b-rule is used.

For the 5 rule we make a case analysis based on the positions of the sequent occurrences $\Diamond A$ and $\emptyset$ from its conclusion $\Gamma\{\Diamond A\}\{\emptyset\}$ inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have two cases: 1) $\emptyset$ inside $\Lambda\{\ \}$, 2) $\emptyset$ inside Σ. The first case is trivial, in the second we have two subcases: 1) $\Diamond A$ inside $\Lambda\{\ \}$ and 2) $\Diamond A$ inside Σ. Case 2.1 is similar to case 2.2 which is as follows:

$$\mathsf{ctr}\,\dfrac{\mathsf{5}\,\dfrac{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}, \Sigma\{\Diamond A\}\{\Diamond A\}\}}{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}, \Sigma\{\Diamond A\}\{\emptyset\}\}}}{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}\}} \quad \rightsquigarrow \quad \mathsf{5}\,\dfrac{\mathsf{ctr}\,\dfrac{\bar{\mathsf{5}}\,\dfrac{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}, \Sigma\{\Diamond A\}\{\Diamond A\}\}}{\Lambda\{\Sigma\{\Diamond A\}\{\Diamond A\}, \Sigma\{\Diamond A\}\{\Diamond A\}\}}}{\Lambda\{\Sigma\{\Diamond A\}\{\Diamond A\}\}}}{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}\}} \quad .$$

□

3 Soundness and Completeness

To prove soundness and completeness, we first need some preliminary definitions.

Definition 1 (frames, models, validity) A *frame* is a pair $(S, \rightarrow)$ of a nonempty set S of *states* and a binary relation $\rightarrow$ on it. A *model* $\mathcal{M}$ is a triple $(S, \rightarrow, V)$ where $(S, \rightarrow)$ is a frame and V is a a mapping which assigns a subset of S to each proposition, and which is called *valuation*. A model $\mathcal{M}$ as given above induces a relation $\models$ between states and formulas which is defined as usual. In particular we have $s \models p$ iff $s \in V(p)$, $s \models \bar{p}$ iff $s \notin V(p)$, $s \models A \vee B$ iff $s \models A$ or $s \models B$, $s \models A \wedge B$ iff $s \models A$ and $s \models B$, $s \models \Diamond A$ iff there is a state t such that $s \rightarrow t$ and $t \models A$, and $s \models \Box A$ iff for all t if $s \rightarrow t$ then $t \models A$. Further, a formula A is *valid in a model* $\mathcal{M}$, denoted $\mathcal{M} \models A$, if for all states s of $\mathcal{M}$ we have $s \models A$. A formula A is *valid in a frame* $(S, \rightarrow)$, denoted $(S, \rightarrow) \models A$, if for all valuations V we have $(S, \rightarrow, V) \models A$. A formula is *valid* if it is valid in all frames. For a subset $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ we call a frame an X-*frame* if it satisfies all the conditions determined by the names in X. A formula is X-*valid* if it is valid in all X-frames.

The $\mathsf{5}$-rule is requires some care when proving its soundness because it is defined in terms of a two-hole context. We first show how it is derivable for three rules which, modulo built-in contraction, are special cases of the $\mathsf{5}$-rule. The soundness of these rules is then easy to establish.

Lemma 2 *The* $\mathsf{5}$*-rule is derivable for* $\{\mathsf{5a}, \mathsf{5b}, \mathsf{5c}, \mathsf{ctr}\}$*, where* $\mathsf{5a}$,$\mathsf{5b}$,$\mathsf{5c}$ *are the rules*

$$\mathsf{5a}\,\frac{\Gamma\{[\Delta], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}} \quad , \quad \mathsf{5b}\,\frac{\Gamma\{[\Delta], [\Lambda, \Diamond A]\}}{\Gamma\{[\Delta, \Diamond A], [\Lambda]\}} \quad , \quad \mathsf{5c}\,\frac{\Gamma\{[\Delta, [\Lambda, \Diamond A]]\}}{\Gamma\{[\Delta, \Diamond A, [\Lambda]]\}} \quad .$$

Proof Seen bottom-up, the $\mathsf{5}$-rule allows to put a formula $\Diamond A$ which occurs at a node different from the root into an arbitrary node. We can use contraction to duplicate $\Diamond A$ and move one copy to the root and also to some child of the root by $\mathsf{5a}$. By $\mathsf{5b}$ we can move it to any child of the root and by $\mathsf{5c}$ into any descendant of a child of the root. □

Lemma 3 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*,* $\Gamma\{\ \}$ *be a context and* A, B *be formulas. If the formula* $A \supset B$ *is* X*-valid then* $\Gamma\{A\} \supset \Gamma\{B\}$ *is* X*-valid.*

Proof By induction on the depth of $\Gamma\{\ \}$. We use the soundness of some Hilbert-style axiomatisation of K+X. To show the validity of

$$(\Gamma_1, [\Gamma_2\{A\}]) \supset (\Gamma_1, [\Gamma_2\{B\}])$$

we use the induction hypothesis to get $\Gamma_2\{A\} \supset \Gamma_2\{B\}$, necessitation to get $\Box(\Gamma_2\{A\} \supset \Gamma_2\{B\})$, the k-axiom to get $\Box(\Gamma_2\{A\}) \supset \Box(\Gamma_2\{B\})$, and finally propositional reasoning to get $\Gamma_1, [\Gamma_2\{A\}] \supset \Gamma_1, [\Gamma_2\{B\}]$. □

Theorem 1 **(Soundness)** *Let* Γ, Δ *and* $\Gamma_1, \ldots, \Gamma_n$ *be sequents and let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *Then the following hold:*

(i) For any rule $\rho \in \mathsf{K}$ *if* $\rho\,\dfrac{\Gamma_1\ \ldots\ \Gamma_n}{\Delta}$ *then* $\Gamma_1 \wedge \cdots \wedge \Gamma_n \supset \Delta$ *is valid.*

(ii) For any rule $\rho \in \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *if* $\rho\,\dfrac{\Gamma}{\Delta}$ *then* $\Gamma \supset \Delta$ *is* $\{\rho\}$*-valid.*

(iii) If $\mathsf{K} + \mathsf{X} \vdash \Gamma$ *then* Γ *is* X*-valid.*

Proof The axiom is valid in all frames which follows from an induction on $\Gamma\{\ \}$ where necessitation is used in the induction step. Thus (i) and (ii) imply (iii). Most cases of (i) are trivial, for the $\wedge$-rule it follows from an induction on the context and uses the implication $\Box A \wedge \Box B \supset \Box(A \wedge B)$. Lemma 3 used together with the k-axiom yields that the premise of the k-rule implies its conclusion. The cases from (ii) for the $\{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}\}$-rules are similar to the k-rule, using the corresponding modal axiom and for the corresponding frames.

For the soundness of the $\mathsf{5}$-rule we use Lemma 2 and show soundness of the rules $\mathsf{5a}, \mathsf{5b}, \mathsf{5c}$. For $\mathsf{5c}$ we show that a euclidean countermodel for the conclusion is also a countermodel for the premise, the other cases are similar. A countermodel for $[\Delta, \Diamond A, [\Lambda]]$ has to contain states $s \rightarrow t \rightarrow u$ such that $t \not\models \Delta$, $u \not\models \Lambda$ and $v \not\models A$ for any v with $t \rightarrow v$. We need to show that for any w with $u \rightarrow w$ we have $w \not\models A$. By euclideanness we obtain, in this order: $t \rightarrow t, u \rightarrow t, t \rightarrow w$. Thus $w \not\models A$. □

Completeness. In order to prove completeness, we need some preliminary definitions which will help us to extract a tree-like Kripke model from a sequent.

Definition 2 (subtree of a sequent) A sequent Δ is an *immediate subtree* of a sequent Γ if there is a sequent Λ such that $\Gamma = \Lambda, [\Delta]$. It is a *proper subtree* if it is an immediate subtree either of Γ or of a proper subtree of Γ, and it is a *subtree* if it is either a proper subtree of Γ or $\Delta = \Gamma$. The set of all subtrees of Γ is denoted by $st(\Gamma)$. A formula A is *in* a sequent Γ if $A \in \Gamma$ and it is *inside* Γ if there is a subtree Δ of Γ such that $A \in \Delta$.

Our sequents are based on multisets. We need a way to stop proof search once their underlying sets remain the same, so we need the following notion:

Definition 3 The *set sequent* of the sequent

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

is the underlying set of

$$A_1, \ldots, A_m, [\Lambda_1], \ldots, [\Lambda_n] \quad ,$$

where $\Lambda_1 \ldots \Lambda_n$ are the set sequents of $\Delta_1 \ldots \Delta_n$. Clearly the set sequent of a given sequent is again a sequent since a set is a multiset.

We will not directly prove completeness of the systems $\mathsf{K} + \mathsf{X}$, but of different, equivalent systems $(\mathsf{K} + \mathsf{X})^\circ$ that we define now. For each rule ρ we define a rule ρ° which keeps the main formula from the conclusion. For most rules $\rho = \rho^\circ$ except for the following rules:

$$\wedge^\circ \frac{\Gamma\{A \wedge B, A\} \quad \Gamma\{A \wedge B, B\}}{\Gamma\{A \wedge B\}} \qquad \vee^\circ \frac{\Gamma\{A \vee B, A, B\}}{\Gamma\{A \vee B\}}$$

$\Box^\circ \dfrac{\Gamma\{\Box A, [A]\}}{\Gamma\{\Box A\}}$ where in the conclusion the node of the active formula does not have a child node which contains A

$\mathsf{d}^\circ \dfrac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}}$ where in the conclusion the node of the active formula does not have a child node.

In addition, each rule ρ° carries the proviso that for all of its premises the set sequent is different from the set sequent of the conclusion. Given a system $\mathcal{S} \subseteq \{\wedge, \vee, \Box, \mathsf{k}, \mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ the system $\mathcal{S}^\circ$ is obtained by replacing each rule $\rho \in \mathcal{S}$ by ρ°. Systems $\mathcal{S}$ and $\mathcal{S}^\circ$ will turn out to be equivalent, as we will know after the completeness theorem. For now we just prove one direction of the equivalence.

Lemma 4 *For all systems* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *and for all sequents* Γ *we have that* $(\mathsf{K} + \mathsf{X})^\circ \vdash \Gamma$ *implies* $(\mathsf{K} + \mathsf{X}) \vdash \Gamma$.

Proof By a standard induction on the proof tree, using contraction and weakening admissibility for $\mathsf{K} + \mathsf{X}$. □

In order to prove completeness we need some closures of relations.

Definition 4 Let $\rightarrow$ be a binary relation on a set S. Then $\leftarrow$ denotes its inverse, $\leftrightarrow$ its symmetric closure, $\rightarrow^+$ its transitive closure and $\rightarrow^*$ its reflexive-transitive closure. For $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ $\rightarrow^{\mathsf{X}}$ denotes the smallest relation that includes $\rightarrow$ and has the properties in X. The same conventions are used for different arrows that denote relations, such as $\Rightarrow$, the inverse of which is $\Leftarrow$, and so on.

We will see shortly that $\to^{\mathsf{X}}$ is well-defined. First we need to characterise the euclidean and the transitive-euclidean closure of a relation.

Definition 5 Let $\to$ be a binary relation on a set S and let $s, t \in S$. A *euclidean connection* for $\to$ from s to t is a nonempty sequence $s_1 \dots s_n$ of elements of S such that we have

$$s \leftarrow s_1 \leftrightarrow s_2 \leftrightarrow \dots \leftrightarrow s_n \to t \qquad .$$

A *transitive-euclidean connection* is defined likewise but such that

$$s = s_1 \leftrightarrow s_2 \leftrightarrow \dots \leftrightarrow s_n \to t \qquad .$$

We write $s \to_{(4)5} t$ if there is a (transitive-)euclidean connection for $\to$ from s to t.

Lemma 5 *Let $\to$ be a binary relation on a set S. Then the following hold:*
(i) For all $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ the relation $\to^{\mathsf{X}}$ is well-defined.
(ii) The relation $\to \cup \to_5$ is the least euclidean relation that contains $\to$.
(iii) The relation $\to_{45}$ is the least transitive and euclidean relation that contains $\to$.

Proof (i) is easy to check except for the cases for $\{\mathsf{5}\}$ and $\{\mathsf{4}, \mathsf{5}\}$, which follow from (ii) and (iii).

(ii) Euclideanness is easy to check. For leastness we show that any euclidean relation $\Rightarrow$ that includes $\to$ also includes $\to_5$. If $s \to_5 t$ then $s \Rightarrow_5 t$. We show $s \Rightarrow_5 t$ for a euclidean connection of length n implies $s \Rightarrow t$ by induction on n. Assume there is an s_i in the euclidean connection such that $s_{i-1} \Rightarrow s_i \Leftarrow s_{i+1}$. Then we have two smaller euclidean connections to which we apply the induction hypothesis and obtain $s \Rightarrow t$ by euclideanness. If there is no such s_i then the euclidean connection looks as follows:

$$s = s_0 \Leftarrow s_1 \Leftarrow \dots \Leftarrow s_j \Rightarrow \dots \Rightarrow s_n \Rightarrow s_{n+1} = t \quad ,$$

and by euclideanness we have $s_{j-1} \Rightarrow s_{j+1}$ and thus removing s_j yields a smaller euclidean connection from s to t which by induction hypothesis implies $s \Rightarrow t$.

(iii) Euclideanness and transitivity are easy to check. For leastness we show that any transitive-euclidean relation $\Rightarrow$ that includes $\to$ also includes $\to_{45}$. If $s \to_{45} t$ then $s \Rightarrow_{45} t$. If there is no s_i in the transitive-euclidean such that $s_i \Leftarrow s_{i+1}$, then $s \Rightarrow t$ follows by transitivity. Otherwise, choose the first such s_i. We have a euclidean connection from s_i to t, thus similarly to (ii) obtain $s_i \Rightarrow t$ and by transitivity $s \Rightarrow s_i$ and $s \Rightarrow t$. □

Definition 6 Let $\to$ be a binary relation on a set S. Its *serial closure*, denoted $\to^{\mathsf{d}}$, is obtained from $\to$ by adding $s \to s$ for each $s \in S$ which violates seriality. For $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ the relation $\to^{\mathsf{X} \cup \{\mathsf{d}\}}$ is defined as $(\to^{\mathsf{X}})^{\mathsf{d}}$.

Lemma 6 *Let $\to$ be a binary relation on a set S. If $\to$ satisfies a frame condition in $\{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ then $\to^{\mathsf{d}}$ also satisfies that frame condition.*

Proof For reflexivity this is clear since a reflexive relation is its own serial closure. For symmetry this is clear since only loops are added, which are their own inverses. For transitivity, assume that we have $s \to^{\mathsf{d}} t$ and $t \to^{\mathsf{d}} u$. If either $s = t$ or $t = u$ then we have $s \to^{\mathsf{d}} u$. So assume $s \neq t$ and $t \neq u$. Then $s \to t$ and $t \to u$ and by transitivity of $\to$ we get $s \to u$ and thus $s \to^{\mathsf{d}} u$.

For euclideanness, assume that $s \to^{\mathsf{d}} t$ and $s \to^{\mathsf{d}} u$. We need to show that $t \to^{\mathsf{d}} u$. If $s = t$ then we are done, so assume $s \neq t$ which implies $s \to t$. Since $s \to^{\mathsf{d}} u$ and since s does not violate seriality we have $s \to u$. By euclideanness of $\to$ we obtain $t \to u$ and thus $t \to^{\mathsf{d}} u$. □

Repeat
(step 1) Keep applying the rules in $((\mathsf{K}+\mathsf{X})\setminus\{\Box,\mathsf{d}\})^\circ$ as long as possible.
(step 2) Wherever possible, apply the rules in $((\mathsf{K}+\mathsf{X})\cap\{\Box,\mathsf{d}\})^\circ$ once.
Until each non-axiomatic derivation leaf is finished.

Fig. 5 The algorithm $\mathit{prove}(\Gamma,\mathsf{X})$

Definition 7 (cyclic, finished, $\mathit{prove}(\Gamma,\mathsf{X})$**)** A leaf of a sequent is *cyclic* if there is an inner node in the sequent that carries the same set of formulas. A node in a sequent is *finished* for a system $\mathcal{S}$ if no rule from $\mathcal{S}$ applies to a formula in this node. A sequent is *finished* for a system $\mathcal{S}$ if all its nodes are either finished for $\mathcal{S}$ or cyclic. We define a procedure $\mathit{prove}(\Gamma,\mathsf{X})$, which takes a sequent Γ and a system $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ and builds a derivation tree for Γ by applying rules from $(\mathsf{K}+\mathsf{X})^\circ$ to non-axiomatic and unfinished derivation leaves in a bottom-up fashion. It is shown in Figure 5. If $\mathit{prove}(\Gamma,\mathsf{X})$ terminates and all derivation leaves are axiomatic then it *succeeds* and if it terminates and there is a non-axiomatic derivation leaf then it *fails*.

Definition 8 The *size* of a sequent is the number of nodes of its corresponding tree. The set of subformulas of a sequent Γ, denoted $\mathit{sf}(\Gamma)$ is the set of all subformulas of all formulas which are element of some node of the sequent.

Lemma 7 (Termination) *For all sets* $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *and for all sequents* Γ *the procedure* $\mathit{prove}(\Gamma,\mathsf{X})$ *terminates after at most* $2^{|\mathit{sf}(\Gamma)|}$ *iterations (of the repeat-until-loop).*

Proof Consider a sequence of sequents along a given branch of the derivation starting from the root. A rule application in step 1 does not create new nodes in the sequent and causes the set of formulas at some node in the sequent to strictly grow. By the subformula property only finitely many formulas can occur in a node, so step 1 terminates. If after step 1 there is an unfinished leaf in a sequent then the size of the sequent strictly grows in step 2. Since there are only $2^{|\mathit{sf}(\Gamma)|}$ different sets of formulas that can occur each unfinished sequent leaf has to be cyclic eventually. □

The current set of modal rules does not allow a modular completeness result of the form "if Γ is X-valid then $\mathsf{K}+\mathsf{X}\vdash\Gamma$". In particular it is easy to check that we have

Fact 2 *For some formula* A *we have*
(i) $\mathsf{K}+\{\mathsf{t},\mathsf{5}\}\nvdash\Box A\supset\Box\Box A$ *and*
(ii) $\mathsf{K}+\{\mathsf{b},\mathsf{4}\}\nvdash\Diamond A\supset\Box\Diamond A$.

However, while not every combination of modal rules is sound and complete for the respective set of frames, at least for each set of frames which can be characterised by our five axioms there is a combination of modal rules which is sound and complete.

Definition 9 Let $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$. The set X is *45-complete* if for $\rho\in\{\mathsf{4},\mathsf{5}\}$ we have that if all X-frames satisfy ρ then $\rho\in\mathsf{X}$.

Both of the sets $\{\mathsf{t},\mathsf{5}\}$ and $\{\mathsf{b},\mathsf{4}\}$ are not 45-complete, for example, while both $\{\mathsf{t},\mathsf{4},\mathsf{5}\}$ and $\{\mathsf{b},\mathsf{4},\mathsf{5}\}$ are. Our completeness result will hold for 45-complete X.

Theorem 3 (Completeness) *For all 45-complete sets* $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *and for all sequents* Γ *the following hold:*
(i) If Γ *is* X*-valid then* $\mathsf{K}+\mathsf{X}\vdash\Gamma$*.*
(ii) If $\mathit{prove}(\Gamma,\mathsf{X})$ *fails then* Γ *is not* X*-valid.*

Proof The contrapositive of (i) follows from (ii): if $\mathsf{K}+\mathsf{X} \nvdash \Gamma$ then by Lemma 4 also $(\mathsf{K}+\mathsf{X})^\circ \nvdash \Gamma$ and thus in particular $prove(\Gamma, \mathsf{X})$ cannot yield a proof and by Lemma 7 has to fail. For (ii) we define a model $\mathcal{M}$ on an X-frame for which we prove that it is a countermodel for Γ. Let Γ^* be the set sequent of the non-axiomatic finished sequent obtained. Let Y be the set of all cyclic leaves in Γ^*. Let $S = st(\Gamma^*) \setminus Y$. Let $f : Y \to S$ be some function which maps a cyclic leaf to a sequent in S whose root carries the same set of formulas and extend f to $st(\Gamma^*)$ by the identity on S. Define a binary relation $\to$ on S such that $\Delta \to \Lambda$ iff either 1) Λ is an immediate subtree of Δ or 2) Δ has an immediate subtree $\Sigma \in Y$ and $f(\Sigma) = \Lambda$. Let $V(p) = \{\Delta \in S \,|\, \bar{p} \in \Delta\}$. Let $\mathcal{M} = (S, \to^{\mathsf{X}}, V)$. We prove three claims about $\mathcal{M}$, each claim depending on the next. Since all rules seen top-down preserve countermodels Claim 1 implies that $\mathcal{M} \not\models \Gamma$.

Claim 1 For each sequent $\Delta \in st(\Gamma^*)$ we have that $\mathcal{M}, f(\Delta) \not\models \Delta$.

By induction on the depth of Δ. For depth zero this follows from Claim 2 and the fact that a formula is in Δ iff it is in $f(\Delta)$. So let

$$\Delta = A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n] \quad \text{and} \quad n > 0 \quad .$$

Then $f(\Delta) = \Delta$. We have $\mathcal{M}, f(\Delta) \not\models A_i$ for all $i \leq m$ by Claim 2 and $\mathcal{M}, \Delta \not\models [\Delta_i]$ because $\Delta \to f(\Delta_i)$ and by induction hypothesis $\mathcal{M}, f(\Delta_i) \not\models \Delta_i$.

Claim 2 For each sequent $\Delta \in S$ and for each formula $A \in \Delta$ we have that $\mathcal{M}, \Delta \not\models A$.

By induction on the depth of A. For atoms it is clear from the definition of $\mathcal{M}$ and since Γ^* is not axiomatic. For the propositional connectives it is clear from the shape of the $\wedge, \vee$-rules. If $A = \Box B$ then by the $\Box$-rule we have some $[\Lambda] \in \Delta$ with $B \in \Lambda$. By induction hypothesis we have $\mathcal{M}, \Lambda \not\models B$ and thus $\mathcal{M}, \Delta \not\models \Box B$. If $A = \Diamond B$ then by Claim 3 we have $B \in \Lambda$ for all Λ with $\Delta \to^{\mathsf{X}} \Lambda$, and thus $\mathcal{M}, \Lambda \not\models B$. Thus $\mathcal{M}, \Delta \not\models \Diamond B$.

Claim 3 For all sequents $\Delta, \Lambda \in S$ with $\Delta \to^{\mathsf{X}} \Lambda$ and for each formula A it holds that if $\Diamond A \in \Delta$ then $A \in \Lambda$.

We make a case analysis on X. Note that each modal logic has exactly one 45-complete axiomatisation, with the exception of $\mathsf{S5}$, which has two.

K $\mathsf{X} = \emptyset$: By the definition of $\to$ there is an immediate subtree of Δ whose root node carries the same set of formulas as the root node of Λ. By the k-rule we have A in (the root node of) all immediate subtrees of Δ.

T $\mathsf{X} = \{\mathsf{t}\}$: $\Delta \to^{\{\mathsf{t}\}} \Lambda$ iff $\Delta \to \Lambda$ or $\Delta = \Lambda$. In the second case $A \in \Lambda$ follows from the t-rule.

KB $\mathsf{X} = \{\mathsf{b}\}$: $\Delta \to^{\{\mathsf{b}\}} \Lambda$ iff $\Delta \to \Lambda$ or $\Lambda \to \Delta$. In the second case $A \in \Lambda$ follows by the b-rule.

K4 $\mathsf{X} = \{\mathsf{4}\}$: $\Delta \to^{\{\mathsf{4}\}} \Lambda$ iff there is a sequence

$$\Delta = \Delta_0 \to \Delta_1 \to \Delta_2 \to \cdots \to \Delta_n = \Lambda \ ,$$

with $n \geq 1$. An induction on i gives us that $\Diamond A \in \Delta_i$ for $0 \leq i \leq n$ by using the $\mathsf{4}$-rule. By the k-rule it follows that $A \in \Delta_n$.

K5 $\mathsf{X} = \{\mathsf{5}\}$: By Lemma 5 we have $\Delta \to^{\{\mathsf{5}\}} \Lambda$ iff $\Delta \to \Lambda$ or there is a euclidean connection from Δ to Λ. In the second case there are sequents Π, Σ such that $\Delta \leftarrow \Pi$ and $\Sigma \to \Lambda$. Thus there is an immediate subtree Δ' of Π with the same formulas as Δ and an immediate subtree Λ' of Σ with the same formulas as Λ. Since $\Diamond A \in \Delta$ we have $\Diamond A \in \Delta'$ and since $\Delta' \neq \Gamma^*$ by the $\mathsf{5}$-rule we have $\Diamond A \in \Sigma$. Thus by the k-rule we have A in Λ' and thus in Λ.

K45 $\mathsf{X} = \{4, 5\}$: By Lemma 5 we have $\Delta \to^{\{4,5\}} \Lambda$ iff $\Delta \to \Lambda$ or there is a transitive-euclidean connection from Δ to Λ. In the second case there is a sequent Σ such that $\Sigma \to \Lambda$ and thus an immediate subtree Λ' of Σ with the same formulas as Λ. Since $\Diamond A \in \Delta$, by the $\mathsf{5}$- and $\mathsf{4}$-rules we have $\Diamond A$ in every subtree of Γ^* and thus also in Σ, and by the k-rule we have A in Λ' and thus in Λ. (It is sufficient to have the $\mathsf{5a}$-rule instead of the $\mathsf{5}$-rule for all X which contain $\mathsf{4}$.)

KB5 $\mathsf{X} = \{\mathsf{b}, 4, 5\}$: $\Delta \to^{\{\mathsf{b},4,5\}} \Lambda$ iff $\Delta \leftrightarrow^+ \Lambda$. Thus there is a sequent Σ such that either $\Sigma \to \Lambda$ or $\Sigma \leftarrow \Lambda$. Rule $4, 5$ imply that $\Diamond A$ is in every subtree of Γ^* and thus in particular in Σ. We have $A \in \Lambda$ in the first case by the k-rule and in the second case by the b-rule.

KTB $\mathsf{X} = \{\mathsf{b}, \mathsf{t}\}$: $\Delta \to^{\{\mathsf{b},\mathsf{t}\}} \Lambda$ iff $\Delta \to \Lambda$ or $\Delta \leftarrow \Lambda$ or $\Delta = \Lambda$. In these cases $A \in \Lambda$ respectively follows from the k- or b- or t-rule.

S4 $\mathsf{X} = \{\mathsf{t}, 4\}$: $\Delta \to^{\{\mathsf{t},4\}} \Lambda$ iff $\Delta \to^+ \Lambda$ or $\Delta = \Lambda$. In the first case $A \in \Lambda$ follows from the rules $\mathsf{4}$ and k and in the second case from the t-rule.

S5(1) $\mathsf{X} = \{\mathsf{t}, 4, 5\}$: $\Delta \to^{\{\mathsf{t},4,5\}} \Lambda$ iff $\Delta \leftrightarrow^* \Lambda$. We have $\Diamond A$ in all subtrees of Γ^* by the rules $4, 5$ and thus also A by the t-rule.

S5(2) $\mathsf{X} = \{\mathsf{d}, \mathsf{b}, 4, 5\}$: $\Delta \to^{\{\mathsf{d},\mathsf{b},4,5\}} \Lambda$ iff $\Delta \leftrightarrow^* \Lambda$. We have $\Diamond A$ in all subtrees of Γ^* by the rules $4, 5$ and thus also $\Diamond A \in \Lambda$. By the d-rule the root of Λ has a child node. By the $\mathsf{4}$-rule $\Diamond A$ is in this child node and by the b-rule $A \in \Lambda$.

KD,KDB,KD4,KD5,KD45 The argument for all these cases is similar to the same system without d. Take the corresponding X, then $\Delta \to^{\mathsf{X} \cup \{\mathsf{d}\}} \Lambda$ iff $\Delta \to^{\mathsf{X}} \Lambda$ or ($\Delta = \Lambda$ and there is no Δ' with $\Delta \to^{\mathsf{X}} \Delta'$). In the second case, due to the d-rule, there is no formula $\Diamond A$ in Δ and thus our claim is trivially true. □

By the termination lemma, soundness and part (ii) of the completeness theorem we get decidability:

Corollary 1 (Decidability) *For all* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ *it is decidable whether a sequent is* X-*valid.*

4 Syntactic Cut Elimination

While cut admissibility is an easy corollary of the completeness theorem, it is still interesting to provide a nontrivial procedure which removes cuts from a proof. The existence of a step-by-step cut elimination procedure shows a certain symmetry, a certain good design of the inference rules. Also, it can serve as a starting point for a computational interpretation, maybe along the lines of [12].

We now see a cut elimination procedure which follows the lines of the one for system $\mathsf{G3}$ for first-order predicate logic, see for example [19]. The interesting twist is that the modalities require some form of multicut, similar to Gentzen's original procedure, even though contraction is admissible.

Definition 10 The *depth* of a formula A, denoted *depth*(A), is defined as usual:

$$\begin{aligned}
&depth(p) = depth(\bar{p}) = 0\\
&depth(\Box A) = depth(\Diamond A) = depth(A) + 1\\
&depth(A \wedge B) = depth(A \vee B) = max(depth(A), depth(B)) + 1 \quad .
\end{aligned}$$

$$\text{cut}\ \frac{\Gamma\{A\} \qquad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

Fig. 6 The cut rule

$$\dot{\mathsf{d}}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \dot{\mathsf{t}}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \dot{\mathsf{b}}\ \frac{\Gamma\{[\Delta,[\Sigma]]\}}{\Gamma\{\Sigma,[\Delta]\}}$$

$$\dot{4}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{[[\Delta]]\}} \qquad \dot{5}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}\ \ depth(\Gamma\{\ \}\{[\Delta]\}) > 0$$

Fig. 7 Modal structural rules

Definition 11 Given an instance of the cut rule as shown in Figure 6, its *cut formula* is A and its *cut rank* is one plus the depth of its cut formula. For $r \geq 0$ we define the rule cut_r which is cut with at most rank r. The *cut rank* of a derivation is the supremum of the cut ranks of its cuts. A rule is *cut-rank (and depth-) preserving admissible* for a system $\mathcal{S}$ if for all $r \geq 0$ the rule is (depth-preserving) admissible for $\mathcal{S} + \mathsf{cut}_r$. A rule is *cut-rank (and depth-) preserving invertible* for a system $\mathcal{S}$ if its inverse is *cut-rank (and depth-) preserving admissible* for $\mathcal{S}$.

Definition 12 Let $\{\Delta\}^n$ denote $\underbrace{\{\Delta\}\ldots\{\Delta\}}_{n-\text{times}}$. For $\mathsf{Y} \subseteq \{4,5\}$ and $n \geq 0$ we define the rule

$$\mathsf{Y}\text{-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \qquad \Gamma\{\Diamond\bar{A}\}\{\Diamond\bar{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

with the proviso that there is a derivation from $\Gamma\{\Diamond\bar{A}\}\{\Diamond\bar{A}\}^n$ to $\Gamma\{\Diamond\bar{A}\}\{\emptyset\}^n$ in system Y.

Fact 4 *Consider an instance of* Y-cut *as above.*
If $\mathsf{Y} = \emptyset$ *then it is an instance of* cut*, i.e.* $n = 0$.
If $\mathsf{Y} = \{4\}$ *then* $\Gamma\{\ \}\{\ \}^n$ *is of the form* $\Gamma_1\{\{\ \},\Gamma_2\{\ \}^n\}$.
If $\mathsf{Y} = \{5\}$ *then the first hole is inside a box, i.e.* $depth(\Gamma\{\ \}\{\emptyset\}^n) > 0$.
(If $\mathsf{Y} = \{4,5\}$ *then nothing can be said about the context since the proviso is trivially fulfilled.)*

The rules which are shown in Figure 7 are called *structural modal rules* for the modal axioms. They are structural in the sense of not affecting connectives of formulas. The modal rules $\mathsf{k}, \mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5$ are all $\Diamond$*-rules*, in the sense that the active formula in the conclusion has $\Diamond$ as main connective. We need the admissibility of these structural modal rules for our cut elimination procedure. They all turn out to be cut-rank preserving admissible, but for $\dot{\mathsf{d}}$ unfortunately we cannot show this unless we have cut elimination. So seriality is a bit special. Our solution is to eliminate cut in the presence of $\dot{\mathsf{d}}$ and only afterwards replace $\dot{\mathsf{d}}$ by d.

The structural modal rules have the obvious corresponding frame conditions. We slightly extend the definition of X-*frame* and *45-completeness* for sets $\mathsf{X} \subseteq \{\mathsf{d}, \dot{\mathsf{d}}, \mathsf{t}, \mathsf{b}, 4, 5\}$ in the obvious way. So if either d or $\dot{\mathsf{d}}$ or both are in X then an X-frame is serial.

Before we eliminate the cut we first need to make sure that contraction and weakening can be eliminated without increasing the cut rank. We just strengthen Lemma 1 accordingly to get the following lemma.

Lemma 8 *For each system* $\mathsf{K}+\mathsf{X}$ *with* $\mathsf{X} \subseteq \{\mathsf{d}, \dot{\mathsf{d}}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *the following hold:*
(i) The rules $\mathsf{nec}, \mathsf{wk}$ *and* ctr *are depth- and cut-rank preserving admissible.*
(ii) All its rules are depth- and cut-rank preserving invertible.

Proof The proof is just like the one for Lemma 1 except that we also consider cut_r and $\dot{\mathsf{d}}$. In proving contraction admissibility there is one more case which is mildly interesting and which is handled as follows:

$$\mathsf{cut}_r \frac{\Gamma\{\Delta\{A\}, \Delta\{\emptyset\}\} \quad \Gamma\{\Delta\{\bar{A}\}, \Delta\{\emptyset\}\}}{\mathsf{ctr}\,\dfrac{\Gamma\{\Delta\{\emptyset\}, \Delta\{\emptyset\}\}}{\Gamma\{\Delta\{\emptyset\}\}}} \quad \rightsquigarrow$$

$$\mathsf{cut}_r \frac{\mathsf{ctr}\,\dfrac{\mathsf{wk}\,\dfrac{\Gamma\{\Delta\{A\}, \Delta\{\emptyset\}\}}{\Gamma\{\Delta\{A\}, \Delta\{A\}\}}}{\Gamma\{\Delta\{A\}\}} \quad \mathsf{ctr}\,\dfrac{\mathsf{wk}\,\dfrac{\Gamma\{\Delta\{\bar{A}\}, \Delta\{\emptyset\}\}}{\Gamma\{\Delta\{\bar{A}\}, \Delta\{\bar{A}\}\}}}{\Gamma\{\Delta\{\bar{A}\}\}}}{\Gamma\{\Delta\{\emptyset\}\}} \quad .$$

□

Lemma 9 (Admissibility of the modal structural rules)
(i) Let X *be a 45-complete subset of* $\{\dot{\mathsf{d}}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *and let* $\rho \in \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*. If* $\rho \in \mathsf{X}$ *then* $\dot{\rho}$ *is cut-rank preserving admissible for* $\mathsf{K}+\mathsf{X}$*.*
(ii) Let X *be a 45-complete subset of* $\{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*. If* $\mathsf{d} \in \mathsf{X}$ *then* $\dot{\mathsf{d}}$ *is admissible for* $\mathsf{K}+\mathsf{X}$*.*

Proof For (i) the proof works by an outer induction on the number of instances of $\dot{\rho}$ in a given proof, eliminating topmost instances first, and an inner induction on the depth of the proof above such a topmost instance. For each rule $\dot{\rho}$ with $\rho \in \mathsf{X}$ we make a case analysis on the rule σ above $\dot{\rho}$. The induction base and the cases where σ is among $\vee, \wedge, \Box, \mathsf{cut}_r, \dot{\mathsf{d}}$ and t are trivial. We use cut-rank preserving admissibility of contraction and weakening provided by Lemma 9 without explicitly mentioning it.

$\dot{\rho} = \dot{\mathsf{t}}$:

$$\dot{\mathsf{t}}\,\frac{\mathsf{k}\,\dfrac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}}{\Gamma\{\Diamond A, \Delta\}} \quad \rightsquigarrow \quad \mathsf{t}\,\frac{\dot{\mathsf{t}}\,\dfrac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, A, \Delta\}}}{\Gamma\{\Diamond A, \Delta\}}$$

The case for $\sigma = \mathsf{b}$ is similar.

$$\dot{\mathsf{t}}\,\frac{\mathsf{4}\,\dfrac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}}{\Gamma\{\Diamond A, \Delta\}} \quad \rightsquigarrow \quad \mathsf{ctr}\,\frac{\dot{\mathsf{t}}\,\dfrac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, \Diamond A, \Delta\}}}{\Gamma\{\Diamond A, \Delta\}}$$

For $\sigma = \mathsf{5}$ the case is trivial unless the diamond formula in its conclusion is at depth 1. Then there are two cases, either the $\mathsf{5}$-rule moves the formula to somewhere outside the box that is removed by $\dot{\mathsf{t}}$ or somewhere inside it. The second case is similar to the first, which is as follows, where ρ^* denotes several applications of ρ:

$$\dot{\mathsf{t}}\,\frac{\mathsf{5}\,\dfrac{[\Diamond A, \Delta], \Sigma\{\Diamond A\}}{[\Diamond A, \Delta], \Sigma\{\emptyset\}}}{\Diamond A, \Delta, \Sigma\{\emptyset\}} \quad \rightsquigarrow \quad \mathsf{4}^*, \mathsf{wk}^*, \mathsf{ctr}^*\,\frac{\dot{\mathsf{t}}\,\dfrac{[\Diamond A, \Delta], \Sigma\{\Diamond A\}}{\Diamond A, \Delta, \Sigma\{\Diamond A\}}}{\Diamond A, \Delta, \Sigma\{\emptyset\}}$$

$\dot{\rho} = \dot{\mathsf{b}}$:

$$\dot{\mathsf{b}}\,\dfrac{\mathsf{k}\,\dfrac{\Gamma\{[\Delta,\Diamond A,[A,\Sigma]]\}}{\Gamma\{[\Delta,\Diamond A,[\Sigma]]\}}}{\Gamma\{\Sigma,[\Delta,\Diamond A]\}} \quad\rightsquigarrow\quad \mathsf{b}\,\dfrac{\dot{\mathsf{b}}\,\dfrac{\Gamma\{[\Delta,\Diamond A,[A,\Sigma]]\}}{\Gamma\{\Sigma,A,[\Delta,\Diamond A]\}}}{\Gamma\{\Sigma,[\Delta,\Diamond A]\}}$$

$$\dot{\mathsf{b}}\,\dfrac{\mathsf{b}\,\dfrac{\Gamma\{[\Delta,A,[\Diamond A,\Sigma]]\}}{\Gamma\{[\Delta,[\Diamond A,\Sigma]]\}}}{\Gamma\{\Diamond A,\Sigma,[\Delta]\}} \quad\rightsquigarrow\quad \mathsf{k}\,\dfrac{\dot{\mathsf{b}}\,\dfrac{\Gamma\{[\Delta,A,[\Diamond A,\Sigma]]\}}{\Gamma\{\Diamond A,\Sigma,[A,\Delta]\}}}{\Gamma\{\Diamond A,\Sigma,[\Delta]\}}$$

$$\dot{\mathsf{b}}\,\dfrac{\mathsf{4}\,\dfrac{\Gamma\{[\Diamond A,\Delta,[\Diamond A,\Sigma]]\}}{\Gamma\{[\Diamond A,\Delta,[\Sigma]]\}}}{\Gamma\{\Sigma,[\Diamond A,\Delta]\}} \quad\rightsquigarrow\quad \mathsf{5}\,\dfrac{\dot{\mathsf{b}}\,\dfrac{\Gamma\{[\Diamond A,\Delta,[\Diamond A,\Sigma]]\}}{\Gamma\{\Sigma,\Diamond A,[\Diamond A,\Delta]\}}}{\Gamma\{\Sigma,[\Diamond A,\Delta]\}}$$

For $\sigma = \mathsf{5}$ the case is trivial unless the diamond formula in its conclusion is at depth 2 and in the inner box in the premise of $\dot{\mathsf{b}}$. Then there are three similar cases of which we just see the following one:

$$\dot{\mathsf{b}}\,\dfrac{\mathsf{5}\,\dfrac{[\Sigma,[\Diamond A,\Delta]],\Gamma\{\Diamond A\}}{[\Sigma,[\Diamond A,\Delta]],\Gamma\{\emptyset\}}}{\Diamond A,\Delta,[\Sigma],\Gamma\{\emptyset\}} \quad\rightsquigarrow\quad \mathsf{4}^*,\mathsf{wk}^*,\mathsf{ctr}^*\,\dfrac{\dot{\mathsf{b}}\,\dfrac{[\Sigma,[\Diamond A,\Delta]],\Gamma\{\Diamond A\}}{\Diamond A,\Delta,[\Sigma],\Gamma\{\Diamond A\}}}{\Diamond A,\Delta,[\Sigma],\Gamma\{\emptyset\}}$$

$\dot\rho = \dot{\mathsf{4}}$:

$$\dot{\mathsf{4}}\,\dfrac{\mathsf{k}\,\dfrac{\Gamma\{\Diamond A,[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}}{\Gamma\{\Diamond A,[[\Delta]]\}} \quad\rightsquigarrow\quad \mathsf{4}\,\dfrac{\mathsf{wk},\mathsf{k}\,\dfrac{\dot{\mathsf{4}}\,\dfrac{\Gamma\{\Diamond A,[A,\Delta]\}}{\Gamma\{\Diamond A,[[A,\Delta]]\}}}{\Gamma\{\Diamond A,[\Diamond A,[\Delta]]\}}}{\Gamma\{\Diamond A,[[\Delta]]\}}$$

The case for $\sigma = \mathsf{4}$ is similar and the case for $\sigma = \mathsf{5}$ is trivial.

$$\dot{\mathsf{4}}\,\dfrac{\mathsf{b}\,\dfrac{\Gamma\{A,[\Diamond A,\Delta]\}}{\Gamma\{[\Diamond A,\Delta]\}}}{\Gamma\{[[\Diamond A,\Delta]]\}} \quad\rightsquigarrow\quad \mathsf{5}\,\dfrac{\mathsf{wk},\mathsf{b}\,\dfrac{\dot{\mathsf{4}}\,\dfrac{\Gamma\{A,[\Diamond A,\Delta]\}}{\Gamma\{A,[[\Diamond A,\Delta]]\}}}{\Gamma\{[\Diamond A,[\Diamond A,\Delta]]\}}}{\Gamma\{[[\Diamond A,\Delta]]\}}$$

$\dot\rho = \dot{\mathsf{5}}$:

$$\dot{\mathsf{5}}\,\dfrac{\mathsf{k}\,\dfrac{\Gamma\{\Diamond A,[A,\Delta]\}\{\emptyset\}}{\Gamma\{\Diamond A,[\Delta]\}\{\emptyset\}}}{\Gamma\{\Diamond A\}\{[\Delta]\}} \quad\rightsquigarrow\quad \mathsf{5}\,\dfrac{\mathsf{wk},\mathsf{k}\,\dfrac{\dot{\mathsf{5}}\,\dfrac{\Gamma\{\Diamond A,[A,\Delta]\}\{\emptyset\}}{\Gamma\{\Diamond A\}\{[A,\Delta]\}}}{\Gamma\{\Diamond A\}\{\Diamond A,[\Delta]\}}}{\Gamma\{\Diamond A\}\{[\Delta]\}}$$

The case for $\sigma = \mathsf{4}$ is similar and the case for $\sigma = \mathsf{5}$ is trivial. For $\sigma = \mathsf{b}$ we have:

$$\dot{\mathsf{5}}\,\dfrac{\mathsf{b}\,\dfrac{\Gamma\{[A,[\Diamond A,\Delta],\Sigma]\}\{\emptyset\}}{\Gamma\{[[\Diamond A,\Delta],\Sigma]\}\{\emptyset\}}}{\Gamma\{[\Sigma]\}\{[\Diamond A,\Delta]\}} \quad\rightsquigarrow\quad \mathsf{5}\,\dfrac{\mathsf{wk},\mathsf{k}\,\dfrac{\dot{\mathsf{5}}\,\dfrac{\Gamma\{[A,[\Diamond A,\Delta],\Sigma]\}\{\emptyset\}}{\Gamma\{[A,\Sigma]\}\{[\Diamond A,\Delta]\}}}{\Gamma\{\Diamond A,[\Sigma]\}\{[\Diamond A,\Delta]\}}}{\Gamma\{[\Sigma]\}\{[\Diamond A,\Delta]\}}$$

The proof for (ii) is similar to the the one for (i), except that we exclude $\sigma = \mathsf{cut}_r$. The case $\sigma = \mathsf{b}$ is trivial.

$\dot\rho = \dot{\mathsf{d}}$:

$$\dot{\mathsf{d}}\,\dfrac{\mathsf{k}\,\dfrac{\Gamma\{\Diamond A,[A]\}}{\Gamma\{\Diamond A,[\emptyset]\}}}{\Gamma\{\Diamond A\}} \quad\rightsquigarrow\quad \mathsf{d}\,\dfrac{\Gamma\{\Diamond A,[A]\}}{\Gamma\{\Diamond A\}}$$

$$\dot{\mathsf{d}}\,\dfrac{\mathsf{4}\,\dfrac{\Gamma\{\Diamond A,[\Diamond A]\}}{\Gamma\{\Diamond A,[\emptyset]\}}}{\Gamma\{\Diamond A\}} \quad\leadsto\quad \mathsf{d}\,\dfrac{\mathsf{4}\,\dfrac{\mathsf{wk}\,\dfrac{\Gamma\{\Diamond A,[\Diamond A]\}}{\Gamma\{\Diamond A,[\Diamond A,A]\}}}{\Gamma\{\Diamond A,[A]\}}}{\Gamma\{\Diamond A\}}$$

$$\dot{\mathsf{d}}\,\dfrac{\mathsf{5}\,\dfrac{\Gamma\{\Diamond A\}\{[\Diamond A]\}}{\Gamma\{\Diamond A\}\{[\emptyset]\}}}{\Gamma\{\Diamond A\}\{\emptyset\}} \quad\leadsto\quad \mathsf{5}\,\dfrac{\mathsf{d}\,\dfrac{\mathsf{5}\,\dfrac{\mathsf{wk}^2\,\dfrac{\Gamma\{\Diamond A\}\{[\Diamond A]\}}{\Gamma\{\Diamond A\}\{\Diamond A,[\Diamond A,A]\}}}{\Gamma\{\Diamond A\}\{\Diamond A,[A]\}}}{\Gamma\{\Diamond A\}\{\Diamond A\}}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

□

To keep the cut elimination procedure short and uniform, we define a structural rule which moves a box inside a sequent from one place to another. Notice that the conditions on the context in the proviso exactly match the conditions in the Y-cut-rule:

Definition 13 (Y-**str-rule**) For $\mathsf{Y} \subseteq \{\mathsf{4}, \mathsf{5}\}$ we define a rule

$$\mathsf{Y\text{-}str}\,\dfrac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}$$

with the proviso that:
if $\mathsf{Y} = \emptyset$ then $\Gamma\{\ \}\{\ \}$ is of the form $\Gamma'\{\{\ \},\{\ \}\}$,
if $\mathsf{Y} = \{\mathsf{4}\}$ then $\Gamma\{\ \}\{\ \}$ is of the form $\Gamma_1\{\{\ \},\Gamma_2\{\ \}\}$, and
if $\mathsf{Y} = \{\mathsf{5}\}$ then $depth(\Gamma\{\ \}\{\emptyset\}) > 0$.
(This means there is no proviso for the case $\mathsf{Y} = \{\mathsf{4}, \mathsf{5}\}$.)

Lemma 10 (**Admissibility of** Y-str) *For 45-complete* $\mathsf{X} \subseteq \{\dot{\mathsf{d}}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *and for* $\mathsf{Y} \subseteq \{\mathsf{4}, \mathsf{5}\}$ *the rule* Y-str *is cut-rank preserving admissible for system* $\mathsf{K} + \mathsf{X}$ *if* $\mathsf{Y} \subseteq \mathsf{X}$.

Proof For $\mathsf{Y} = \emptyset$ that is trivial. For $\mathsf{Y} = \{\mathsf{4}\}$ the rule is derivable as follows:

$$\mathsf{ctr}\,\dfrac{\mathsf{wk}\,\dfrac{\mathsf{wk}^*\,\dfrac{\dot{\mathsf{4}}^*\,\dfrac{\Gamma\{[\Delta],\Sigma\{\emptyset\}\}}{\Gamma\{[\ldots[\Delta]\ldots],\Sigma\{\emptyset\}\}}}{\Gamma\{\Sigma\{[\Delta]\},\Sigma\{\emptyset\}\}}}{\Gamma\{\Sigma\{[\Delta]\},\Sigma\{[\Delta]\}\}}}{\Gamma\{\Sigma\{[\Delta]\}\}} \quad ,$$

and thus admissible by Lemma 8 and Lemma 9. For $\mathsf{Y} = \{\mathsf{5}\}$ the rule coincides with $\dot{\mathsf{5}}$ and is thus admissible by Lemma 9. For $\mathsf{Y} = \{\mathsf{4}, \mathsf{5}\}$ an instance of the rule is either an instance of the Y-str-rule for $\mathsf{Y} = \{\mathsf{4}\}$ or $\mathsf{Y} = \{\mathsf{5}\}$ and thus admissible as in the previous two cases. □

Lemma 11 (**Reduction Lemma**) *Let* X *be a 45-complete subset of* $\{\dot{\mathsf{d}}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, let* Y *be a subset of* $\{\mathsf{4}, \mathsf{5}\} \cap \mathsf{X}$ *and let* $r > 0$ *and* $n \geq 0$.
(i) If there is a proof

$$\mathsf{cut}_{r+1}\,\dfrac{\overset{\pi_1}{\Gamma\{A\}} \qquad \overset{\pi_2}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}}$$

with π_1 *and* π_2 *in* $\mathsf{K}+\mathsf{X}+\mathsf{cut}_r$ *then* $\mathsf{K}+\mathsf{X}+\mathsf{cut}_r \vdash \Gamma\{\emptyset\}$.
(ii) If there is a proof

$$\mathsf{Y}\text{-}\mathsf{cut}_{r+1}\,\frac{\overset{\pi_1}{\Gamma\{\Box A\}\{\emptyset\}^n} \qquad \overset{\pi_2}{\Gamma\{\Diamond \bar{A}\}\{\Diamond \bar{A}\}^n}}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

with π_1 *and* π_2 *in* $\mathsf{K}+\mathsf{X}+\mathsf{cut}_r$ *then* $\mathsf{K}+\mathsf{X}+\mathsf{cut}_r \vdash \Gamma\{\emptyset\}\{\emptyset\}^n$.

Proof We prove (i) and (ii) simultaneously by induction on $|\pi_1|+|\pi_2|$. We perform a case analysis on the two lowermost rules in π_1 and π_2. If one of the two rules is passive and an axiom then $\Gamma\{\emptyset\}$ is axiomatic as well. If one is active and an axiom then we have

$$\mathsf{cut}_{r+1}\,\frac{\Gamma\{a,\bar{a}\} \qquad \overset{\pi_2}{\Gamma\{\bar{a},\bar{a}\}}}{\Gamma\{\bar{a}\}} \quad\rightsquigarrow\quad \mathsf{ctr}\,\frac{\overset{\pi_2}{\Gamma\{\bar{a},\bar{a}\}}}{\Gamma\{\bar{a}\}} \quad .$$

If one rule is passive then we have

$$\mathsf{cut}_{r+1}\,\frac{\overset{\pi_1}{\Gamma\{A\}} \qquad \rho\,\dfrac{\overset{\pi_2}{\Gamma'\{\bar{A}\}}}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}} \quad\rightsquigarrow\quad \rho\,\frac{\mathsf{cut}_{r+1}\,\dfrac{\bar{\rho}\,\dfrac{\overset{\pi_1}{\Gamma\{A\}}}{\Gamma'\{A\}} \qquad \overset{\pi_2}{\Gamma'\{\bar{A}\}}}{\Gamma'\{\emptyset\}}}{\Gamma\{\emptyset\}}$$

for case (i) and similarly for (ii). This leaves the case that both rules are active and not axioms. For (i) we have:

$$\mathsf{cut}_{r+1}\,\frac{\wedge\,\dfrac{\overset{\pi_1}{\Gamma\{B\}} \qquad \overset{\pi_2}{\Gamma\{C\}}}{\Gamma\{B\wedge C\}} \qquad \rho\,\dfrac{\overset{\pi_3}{\Gamma'\{\bar{B},\bar{C}\}}}{\Gamma\{\bar{B}\vee\bar{C}\}}}{\Gamma\{\emptyset\}} \quad\rightsquigarrow$$

$$\mathsf{cut}_r\,\frac{\overset{\pi_1}{\Gamma\{B\}} \qquad \mathsf{cut}_r\,\dfrac{\mathsf{wk}\,\dfrac{\overset{\pi_2}{\Gamma\{C\}}}{\Gamma\{\bar{B},C\}} \qquad \overset{\pi_3}{\Gamma\{\bar{B},\bar{C}\}}}{\Gamma\{\bar{B}\}}}{\Gamma\{\emptyset\}} \quad .$$

Notice that (i) is a special case of (ii) if A has a modality as its main connective. The remaining case is thus (ii) with both rules active and not axioms, and thus on one side the $\Box$-rule and on the other side either k, t or b (the cases for $\mathsf{4}$ and $\mathsf{5}$ are trivial). The case for the k-rule is as follows:

$$\mathsf{Y\text{-}cut}_{r+1}\,\dfrac{\Box\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{[\Delta]\}}}{\Gamma\{\Box A\}\{[\Delta]\}} \qquad \mathsf{k}\,\dfrac{\overset{\pi_2}{\Gamma'\{\Diamond\bar{A}\}\{\Diamond\bar{A},[\bar{A},\Delta']\}}}{\Gamma'\{\Diamond\bar{A}\}\{\Diamond\bar{A},[\Delta']\}}}{\Gamma\{\emptyset\}\{[\Delta]\}} \quad \rightsquigarrow$$

$$\mathsf{cut}_r\,\dfrac{\mathsf{ctr}\,\dfrac{\mathsf{wk}^2\,\dfrac{\mathsf{Y\text{-}str}\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{[A],[\Delta]\}}}{\Gamma\{\emptyset\}\{[A,\Delta],[A,\Delta]\}}}{\Gamma\{\emptyset\}\{[A,\Delta]\}} \qquad \mathsf{Y\text{-}cut}_{r+1}\,\dfrac{\Box,\mathsf{wk}\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{[\Delta]\}}}{\Gamma\{\Box A\}\{[\bar{A},\Delta]\}} \qquad \overset{\pi_2}{\Gamma'\{\Diamond\bar{A}\}\{\Diamond\bar{A},[\bar{A},\Delta']\}}}{\Gamma\{\emptyset\}\{[\bar{A},\Delta]\}}}{\Gamma\{\emptyset\}\{[\Delta]\}} \quad ,$$

where the Y-str-rule is applicable since its condition on the context matches the condition in the Y-cut-rule, the Y-str-rule is cut-rank-preserving admissible by Lemma 10, weakening and contraction are cut-rank-and-depth-preserving admissible by Lemma 8 and the instance of Y-cut can be removed by induction hypothesis. The cases for t and b are as follows:

$$\mathsf{Y\text{-}cut}_{r+1}\,\dfrac{\Box\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{\emptyset\}}}{\Gamma\{\Box A\}\{\emptyset\}} \qquad \mathsf{t}\,\dfrac{\overset{\pi_2}{\Gamma'\{\Diamond\bar{A}\}\{\bar{A}\}}}{\Gamma'\{\Diamond\bar{A}\}\{\Diamond\bar{A}\}}}{\Gamma\{\emptyset\}\{\emptyset\}} \quad \rightsquigarrow$$

$$\mathsf{cut}_r\,\dfrac{\dot{\mathsf{t}}\,\dfrac{\mathsf{Y\text{-}str}\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{\emptyset\}}}{\Gamma\{\emptyset\}\{[A]\}}}{\Gamma\{\emptyset\}\{A\}} \qquad \mathsf{Y\text{-}cut}_{r+1}\,\dfrac{\Box,\mathsf{wk}\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{\emptyset\}}}{\Gamma\{\Box A\}\{\bar{A}\}} \qquad \overset{\pi_2}{\Gamma'\{\Diamond\bar{A}\}\{\bar{A}\}}}{\Gamma\{\emptyset\}\{\bar{A}\}}}{\Gamma\{\emptyset\}\{\emptyset\}}$$

and

$$\mathsf{Y\text{-}cut}_{r+1}\,\dfrac{\Box\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{[\Delta]\}}}{\Gamma\{\Box A\}\{[\Delta]\}} \qquad \mathsf{b}\,\dfrac{\overset{\pi_2}{\Gamma'\{\Diamond\bar{A}\}\{\bar{A},[\Diamond\bar{A},\Delta']\}}}{\Gamma'\{\Diamond\bar{A}\}\{[\Diamond\bar{A},\Delta']\}}}{\Gamma\{\emptyset\}\{[\Delta]\}} \quad \rightsquigarrow$$

$$\mathsf{cut}_r\,\dfrac{\dot{\mathsf{b}}\,\dfrac{\mathsf{Y\text{-}str}\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{[[A],\Delta]\}}}{\Gamma\{\emptyset\}\{A,[\Delta]\}} \qquad \mathsf{Y\text{-}cut}_{r+1}\,\dfrac{\Box,\mathsf{wk}\,\dfrac{\overset{\pi_1}{\Gamma\{[A]\}\{[\Delta]\}}}{\Gamma\{\Box A\}\{\bar{A},[\Delta]\}} \qquad \overset{\pi_2}{\Gamma'\{\Diamond\bar{A}\}\{\bar{A},[\Diamond\bar{A},\Delta']\}}}{\Gamma\{\emptyset\}\{\bar{A},[\Delta]\}}}{\Gamma\{\emptyset\}\{[\Delta]\}} \quad ,$$

In general the Y-cut, seen upwards, introduces several diamond formulas. One of them is special in being in the same position as its dual cut formula in the other premise. In the transformations given above, the active formula of the diamond-rule above the cut is different from that special formula. That is not always the case, of course, but if the two coincide, then the transformations are simpler. $\Box$

Theorem 5 (Cut Elimination) *Let* X *be a 45-complete subset of either (i)* $\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *or (ii)* $\{\dot{\mathsf{d}},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$*. In both cases we have*

$$\textit{If } \mathsf{K}+\mathsf{X}+\mathsf{cut}\vdash\Gamma \textit{ then } \mathsf{K}+\mathsf{X}\vdash\Gamma\;.$$

Proof For case (ii) the theorem follows from a routine induction on the cut-rank of the given proof, the induction step follows by another induction, on the depth of the proof, and uses the reduction lemma in the case of a maximal-rank cut. For case (i) the theorem follows from case (ii), the fact that d is derivable for $\{\mathsf{k},\dot{\mathsf{d}}\}$ and Lemma 9. $\Box$

5 Towards Modularity

Our goal was to give pure proof systems with the good properties of Negri's labelled sequent systems and in one respect we failed. The main shortcoming of our systems is that they are not as modular as labelled systems. They cover the entire modal cube and are systematic in the sense that there is a one-to-one correspondence between the modal rules and the frame conditions. However, unlike labelled systems, they are not modular in the sense that each combination of modal rules is complete for the corresponding class of frames. This forced us to resort to the kludge of formulating the condition of 45-complete systems and proving completeness only for those. I do not see a way of achieving modularity using these systems, i.e. where frame conditions are formalised in $\Diamond$-rules.

However, during the cut elimination procedure we discovered the possibility of forming proof systems not using $\Diamond$-rules but using the structural rules shown in Figure 7 on page 13. I conjecture that these systems are modular. Notice the absence of the word "45-complete":

Conjecture 1 For each sequent Γ and $\mathsf{X}\subseteq\{\dot{\mathsf{d}},\dot{\mathsf{t}},\dot{\mathsf{b}},\dot{\mathsf{4}},\dot{\mathsf{5}}\}$ we have:

$$\mathsf{K}+\mathsf{X}+\{\mathsf{ctr},\mathsf{wk}\}\vdash\Gamma \quad\text{iff}\quad \Gamma \text{ is } \mathsf{X}\text{-valid}.$$

In particular, the examples which showed the incompleteness of systems $\mathsf{K}+\{\mathsf{t},\mathsf{5}\}$ and $\mathsf{K}+\{\mathsf{b},\mathsf{4}\}$ from Fact 2 are provable in systems $\mathsf{K}+\{\dot{\mathsf{t}},\dot{\mathsf{5}}\}$ and $\mathsf{K}+\{\dot{\mathsf{b}},\dot{\mathsf{4}}\}$, respectively:

$$\Box^2\dfrac{\dot{\mathsf{t}}\dfrac{\dot{\mathsf{5}}\dfrac{\mathsf{k}\dfrac{[\Diamond\bar{A},[A,\bar{A}],[\emptyset]]}{[\Diamond\bar{A},[A],[\emptyset]]}}{[\Diamond\bar{A},[[A]]]}}{\Diamond\bar{A},[[A]]}}{\Diamond\bar{A},\Box\Box A}\qquad\text{and}\qquad\Box^2\dfrac{\dot{\mathsf{b}}\dfrac{\dot{\mathsf{4}}\dfrac{\mathsf{k}\dfrac{[[\bar{A},A],\Diamond A]}{[[\bar{A}],\Diamond A]}}{[[[\bar{A}]],\Diamond A]}}{[\bar{A}],[\Diamond A]}}{\Box\bar{A},\Box\Diamond A}\quad.$$

In the above conjecture, contraction and weakening are present just because they are not built-in in the modal structural rules, but of course it is easy to do so and then to drop contraction and weakening.

We can already prove some parts of the above conjecture:

Theorem 6 *For each sequent Γ and each* X *among* $\{\dot{\mathsf{d}}\}, \{\dot{\mathsf{t}}\}, \{\dot{\mathsf{b}}\}$, *and* $\{\dot{\mathsf{4}}\}$ *we have:*

$$\mathsf{K} + \mathsf{X} + \{\mathsf{ctr}, \mathsf{wk}\} \vdash \Gamma \quad \textit{iff} \quad \Gamma \textit{ is } \mathsf{X}\textit{-valid.}$$

Proof For $\mathsf{X} = \{\dot{\mathsf{d}}\}$ this is a trivial consequence of the completeness of $\mathsf{K} + \{\mathsf{d}\}$ and the derivability of d for $\{\mathsf{k}, \dot{\mathsf{d}}\}$. For $\mathsf{X} = \{\dot{\mathsf{t}}\}$ we prove completeness by embedding the standard sequent calculus for KT for example from [19]. The interesting two cases are as follows:

$$\Box\,\dfrac{\begin{array}{c}\pi\\ \Gamma, A\end{array}}{\Diamond\Gamma, \Box A} \quad\rightsquigarrow\quad \Box\,\dfrac{\mathsf{k}^*\,\dfrac{\mathsf{wk}\,\dfrac{\mathsf{nec}\,\dfrac{\begin{array}{c}\pi'\\ \Gamma, A\end{array}}{[\Gamma, A]}}{\Diamond\Gamma, [\Gamma, A]}}{\Diamond\Gamma, [A]}}{\Diamond\Gamma, \Box A} \qquad\qquad \Diamond\,\dfrac{\begin{array}{c}\pi\\ \Gamma, A\end{array}}{\Gamma, \Diamond A} \quad\rightsquigarrow\quad \dot{\mathsf{t}}\,\dfrac{\mathsf{k}\,\dfrac{\mathsf{wk}\,\dfrac{\mathsf{nec}\,\dfrac{\begin{array}{c}\pi'\\ \Gamma, A\end{array}}{[\Gamma, A]}}{[\Gamma, A], \Diamond A}}{[\Gamma], \Diamond A}}{\Gamma, \Diamond A} \quad .$$

For $\mathsf{X} = \{\dot{\mathsf{b}}\}$ we show admissibility of b and use completeness of $\mathsf{K} + \{\mathsf{b}\}$. The turn rule, essentially from [11],

$$\mathsf{turn}\,\dfrac{\Delta, [\Gamma]}{\Gamma, [\Delta]} \qquad ,$$

is derivable for $\{\mathsf{nec}, \dot{\mathsf{b}}\}$ and using it we can derive the b-rule as follows:

$$\mathsf{turn}^*\,\dfrac{\mathsf{k}\,\dfrac{\mathsf{turn}^*\,\dfrac{\Gamma\{A, [\Diamond A, \Delta]\}}{[\Gamma', A], \Diamond A, \Delta}}{[\Gamma'], \Diamond A, \Delta}}{\Gamma\{[\Diamond A, \Delta]\}} \qquad .$$

For $\mathsf{X} = \{\dot{\mathsf{4}}\}$ we see completeness of the system $\mathsf{K} + \{\dot{\mathsf{4}}'\}$ with the rule

$$\dot{\mathsf{4}}'\,\dfrac{\Gamma\{[\Delta, [\Lambda]], [\Lambda]\}}{\Gamma\{[\Delta, [\Lambda]]\}} \qquad ,$$

which is derivable for $\{\dot{\mathsf{4}}, \mathsf{wk}, \mathsf{ctr}\}$. We define system $(\mathsf{K} + \{\dot{\mathsf{4}}'\})^\circ$ just like we did before. Proof search in this system terminates. To see that, we define the modal depth of a formula as the maximum number of modal connectives encountered along some path from the root to the leaves. The modal depth of a sequent is the modal depth of its corresponding formula. No rule, seen bottom-up, increases the modal depth of the sequent. Thus the depth of any sequent occuring in a proof is bounded. By the subformula property the number of formulas which can appear in a certain node is bounded. Thus the size of any sequent occuring in a proof is bounded. Each rule strictly increases either the size of the sequent or the number of formulas at a certain node and thus a sequent will be reached where no rule is applicable. It is now straightforward to extract a transitive countermodel from a failed proof search as in the proof of Theorem 3. □

Unfortunately, the four cases in the above theorem were proved in four different ways and neither way of proving them can be easily made to work for the other cases. A general method to uniformly prove completeness for all combinations of the modal structural rules is subject of current research.

6 Relation to other Formalisms

Relation to Kashima's systems. As I learned after the publication of [4], Kashima introduced sequent systems for some tense logics in [11] which use essentially the same notion of sequent that I use. He attributes this idea both to Sato [15] and to Bull [6]. Kashima treats tense logics formed from K_t by the axioms for reflexivity, transitivity, totality and connectedness. System K is just the modal fragment of his system and the rules for reflexivity and transitivity are also almost the same – the only difference is that my rules include an implicit contraction to make them invertible and contraction admissible. Apart from considering different logics, in addition to Kashima's work, my completeness proof yields decision procedures, and I give a syntactic cut elimination procedure which gives rise to the modal structural rules. The notion of deep sequent is also considered by Poggiolesi who calls it tree-hypersequent but uses a rather different notation. She gives systems for KD and K4 with a syntactic cut elimination procedure in [14].

Relation to the calculus of structures. In the calculus of structures (or *cos*) inference rules are essentially just term rewriting rules which work on formulas. Formulas are considered to be equivalent modulo some equations, such as commutativity and associativity of disjunction. A proof of a formula is a rewriting sequence starting from some constant t for *true* and ending with that formula. A system for propositional logic from from [3] is shown in Figure 8. Systems for modal logics can be obtained by adding rules from Figure 9.

Essentially, all derivations on nested sequents are also cos derivations: they are are simply more restricted since rules do not apply deeply with respect to all connectives, but only with respect to disjunction and box. It is thus trivial to embed our sequent systems into corresponding cos systems. The reverse direction, embedding cos into deep sequents, requires cut, but is also easy.

Let an instance of $\mathsf{5}{\downarrow}$ be an instance of either $\mathsf{5a}{\downarrow}, \mathsf{5b}{\downarrow}$ or $\mathsf{5c}{\downarrow}$. For a set X of rule names append the symbol $\downarrow$ to each name to obtain $\mathsf{X}{\downarrow}$. Let system KSk be system $\mathsf{KS} + \{\mathsf{nec}{\downarrow}, \mathsf{k}{\downarrow}\}$. The following proposition is easily proved, for details see [3]:

Proposition 1 (Relating deep sequent systems and cos) *For all* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, sequents* Γ *and formulas* A *we have that:*
(i) If $\mathsf{K} + \mathsf{X} \vdash \Gamma$ *then* $\mathsf{KSk} + \mathsf{X}{\downarrow} \vdash \Gamma$*.*
(ii) If $\mathsf{KSk} + \mathsf{X}{\downarrow} + \mathit{cos}\text{-}\mathsf{cut} \vdash A$ *then* $\mathsf{K} + \mathsf{X} + \mathsf{cut} \vdash A$*.*

This proposition, together with cut elimination for our deep sequent systems, trivially yields cut elimination for the corresponding cos systems. By (ii) we just translate a cos-proof with cuts into a deep sequent proof with cuts, eliminate the cuts, and translate back to cos by (i).

Corollary 2 *For all 45-complete* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *the cos system* $\mathsf{KSk} + \mathsf{X}{\downarrow}$ *admits cut elimination.*

Relation to labelled systems. The main technical difference between our systems and labelled systems is that the structural level in labelled systems is more general: it can form an arbitrary graph, while nested sequents are always trees. The main conceptual difference, however, is of course that each deep sequent can be read as a modal formula. So they are further removed from semantics, more “syntactic” than labelled sequents. This shows in our completeness proof: we had to establish certain properties

$$\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S\{a, \bar{a}\}} \qquad \mathsf{s}\!\downarrow \frac{S\{A \wedge (B \vee C)\}}{S\{(A \wedge B) \vee C\}} \qquad \mathsf{c}\!\downarrow \frac{S\{A \vee A\}}{S\{A\}} \qquad \mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{A\}}$$

Fig. 8 Propositional logic in the calculus of structures

$$\mathsf{nec}\!\downarrow \frac{S\{\mathsf{t}\}}{S\{\Box\mathsf{t}\}} \qquad \mathsf{k}\!\downarrow \frac{S\{\Box(A \vee B)\}}{S\{\Box A \vee \Diamond B\}}$$

$$\mathsf{d}\!\downarrow \frac{S\{\Box A\}}{S\{\Diamond A\}} \qquad \mathsf{t}\!\downarrow \frac{S\{A\}}{S\{\Diamond A\}} \qquad \mathsf{b}\!\downarrow \frac{S\{A \vee \Box B\}}{S\{\Box(\Diamond A \vee B)\}} \qquad \mathsf{4}\!\downarrow \frac{S\{\Box(A \vee \Diamond B)\}}{S\{\Box A \vee \Diamond B\}}$$

$$\mathsf{5a}\!\downarrow \frac{S\{\Diamond A \vee \Box B\}}{S\{\Box(\Diamond A \vee B)\}} \qquad \mathsf{5b}\!\downarrow \frac{S\{\Box B \vee \Box(\Diamond A \vee C)\}}{S\{\Box(\Diamond A \vee B) \vee \Box C\}} \qquad \mathsf{5c}\!\downarrow \frac{S\{\Box(A \vee \Box(\Diamond B \vee C))\}}{S\{\Box(A \vee \Diamond B \vee \Box C)\}}$$

Fig. 9 Modal rules in the calculus of structures

of, say, the euclidean closure of a relation, which is not needed for labelled systems. There, that relation is part of the system and it is being closed under euclideanness by the appropriate rule. It also shows in our cut elimination procedure: we had to show admissibility of certain rules in order to push the cut over the rules for the frame properties. This, again, is not needed for labelled systems. There the rules for the frame conditions do not affect the cut elimination procedure at all. So, in some sense we had to do more work in proving our systems complete. I hope that this fact will help in using deep sequent systems for interpolation proofs, for which labelled systems do not seem to be well-suited.

Relation to hypersequents. Deep sequents are a natural generalisation of (modal) hypersequents, in allowing arbitrary nestings of boxed disjunctions instead of just a disjunction of boxed disjunctions. I am not aware of hypersequent systems for K5 or B nor of hypersequent systems with invertible rules and contraction admissibility for the modal logics treated here. A notational simplification I enjoy with respect to hypersequent systems is that the two kinds of context in inference rules (sequent context and hypersequent context) are merged into one.

Relation to the display calculus. Display sequents are closely related, in particular the idea of simply allowing the connective $\Box$ as a structural connective is common to display sequents and deep sequents. However, the proof systems are rather different. Loosely speaking, in the display calculus one has to make a formula bubble up to the top by using the structural rules in order to apply a logical rule to it, while in deep sequent systems one can apply the rule on the spot. This leads to deductive systems with drastically fewer rules and shorter derivations. On the other hand the display calculus so far has captured more modal logics than deep sequents and also enjoys a general cut elimination result, which for deep sequents is subject of current research. As with hypersequents, I am not aware of display systems with invertible rules and contraction admissibility.

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References

1. Arnon Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In Wilfrid Hodges, Martin Hyland, Charles Steinhorn, and John Truss, editors, *Logic: from foundations to applications. Proc. Logic Colloquium, Keele, UK, 1993*, pages 1–32. Oxford University Press, New York, 1996.
2. Nuel D. Belnap, Jr. Display logic. *Journal of Philosophical Logic*, 11:375–417, 1982.
3. Kai Brünnler. *Deep Inference and Symmetry in Classical Proofs*. PhD thesis, Technische Universität Dresden, September 2003.
4. Kai Brünnler. Deep sequent systems for modal logic. In Guido Governatori, Ian Hodkinson, and Yde Venema, editors, *Advances in Modal Logic*, volume 6, pages 107–119. College Publications, 2006.
5. Kai Brünnler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347–361. Springer-Verlag, 2001.
6. Robert A. Bull. Cut elimination for propositional dynamic logic without *. *Mathematische Logik und Grundlagen der Mathematik*, 38:85–100, 1992.
7. Jim Garson. Modal logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Stanford University, Spring 2008. `http://plato.stanford.edu/archives/spr2008/entries/logic-modal/`.
8. Alessio Guglielmi. A system of interaction and structure. *ACM Transactions on Computational Logic*, 8(1):1–64, 2007.
9. Alessio Guglielmi and Lutz Straß burger. Non-commutativity and MELL in the calculus of structures. In L. Fribourg, editor, *CSL 2001*, volume 2142 of *Lecture Notes in Computer Science*, pages 54–68. Springer-Verlag, 2001.
10. Robert Hein and Charles Stewart. Purity through unravelling. In Paola Bruscoli, François Lamarche, and Charles Stewart, editors, *Structures and Deduction*, pages 126–143. Technische Universität Dresden, 2005.
11. Ryo Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53:119–135, 1994.
12. Simone Martini and Andrea Masini. A computational interpretation of modal proofs. In H. Wansing, editor, *Proof theory of modal logic*, volume 2 of *Applied logic series*, pages 213–241. Kluwer, 1996.
13. Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5 – 6):507 – 544, 2005.
14. Francesca Poggiolesi. The tree-hypersequent method for modal propositional logic. In Jacek Malinowski David Makinson and Heinrich Wansing, editors, *Towards Mathematical Philosophy*, Trends in Logic, pages 9–30. Springer, 2009.
15. Masahiko Sato. A study of Kripke-type models for some modal logics by Gentzen's sequential method. *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, 13:381–468, 1977.
16. Charles Stewart and Phiniki Stouppa. A systematic proof theory for several modal logics. In Renate Schmidt, Ian Pratt-Hartmann, Mark Reynolds, and Heinrich Wansing, editors, *Advances in Modal Logic*, volume 5 of *King's College Publications*, pages 309–333, 2005.
17. Phiniki Stouppa. A deep inference system for the modal logic S5. *Studia Logica*, 85(2):199–214, 2007.
18. Lutz Straßburger. *Linear Logic and Noncommutativity in the Calculus of Structures*. PhD thesis, Technische Universität Dresden, 2003.
19. Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 1996.
20. Heinrich Wansing. *Displaying Modal Logic*, volume 3 of *Trends in Logic Series*. Kluwer Academic Publishers, Dordrecht, 1998.
21. Heinrich Wansing. Sequent systems for modal logics. In D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic, 2nd edition*, volume 8, pages 61–145. Kluwer, Dordrecht, 2002.