Interaction and Depth against Nondeterminism in Proof Search

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The Calculus of Structures

- generalizes the sequent calculus with deep inference. [Guglielmi, 99]
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- Inference rules can be applied at any depth inside a formula.
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- Inference rules can be applied at any depth inside a formula.

A proof in the sequent system GS1p

\[
\begin{align*}
\vdash a, \bar{a} & \quad \text{Ax} \quad \vdash b, \bar{b} \\
\vdash b \lor \bar{b} & \quad \text{Rv} \\
\vdash a \lor (\bar{a} \land (b \lor \bar{b})) & \quad \text{Rv}
\end{align*}
\]

A proof in system KSg

\[
\begin{align*}
\text{tt} \downarrow \vdash \text{tt} \\
\text{ai} \downarrow \vdash a \lor \bar{a} \\
\vdash a \lor (\bar{a} \land (b \lor \bar{b}))
\end{align*}
\]
The Calculus of Structures

- generalizes the sequent calculus with deep inference. [Guglielmi, 99]
- Inference rules can be applied at any depth inside a formula.

A proof in the sequent system GS1p | A proof in system KSg

\[
\frac{\vdash a, \bar{a}}{\vdash a \lor (\bar{a} \land (b \lor \bar{b}))}
\]

Deep inference brings shorter proofs.

[Polynomial Size Deep-Inference Proofs Instead of Exponential Size Shallow-Inference Proofs, Guglielmi, 2004]
Deep Inference and Resolution

Cut-free sequent calculus does not polynomially simulate popular proof procedures such as resolution, e.g., [Beame, Pitassi,98].
Deep Inference and Resolution

Cut-free sequent calculus does not polynomially simulate popular proof procedures such as resolution, e.g., [Beame, Pitassi, 98].

The resolution rule

\[
\frac{R \land T}{(R \land a) \lor (T \land \bar{a})}
\]

is derivable in the calculus of structures system for classical logic.
Cut-free sequent calculus does not polynomially simulate popular proof procedures such as resolution, e.g., [Beame, Pitassi,98].

The resolution rule

\[
\begin{array}{c}
R \land T \\
ai \downarrow \\
R \land \neg \neg \neg \\
s \\
s \\
R \land T \land (a \lor \neg a) \\
R \land (a \lor (T \land \neg a)) \\
(R \land a) \lor (T \land \neg a)
\end{array}
\]

is derivable in the calculus of structures system for classical logic.
Deep Inference and Resolution

Cut-free sequent calculus does not polynomially simulate popular proof procedures such as resolution, e.g., [Beame, Pitassi, 98].

The resolution rule

\[
\begin{align*}
\text{ai} \downarrow & \quad R \land T \\
\text{s} & \quad R \land T \land (a \lor \bar{a}) \\
\text{s} & \quad R \land (a \lor (T \land \bar{a})) \\
\text{s} & \quad (R \land a) \lor (T \land \bar{a})
\end{align*}
\]

is derivable in the calculus of structures system for classical logic.

\[
\begin{align*}
\text{s} & \quad S((R \lor U) \land T) \\
\text{ai} \downarrow & \quad S\{\tt\} \\
\text{s} & \quad S((R \land T) \lor U) \\
\text{s} & \quad S(a \lor \bar{a})
\end{align*}
\]
However...

Consider the instance of the sequent calculus inference rule:

\[
\frac{\vdash a, \bar{a} \quad \vdash b, \bar{b}}{\vdash a, b, \bar{a} \land \bar{b}} \quad \text{R}^\land
\]

\[\text{R}^\land\]
However...

Consider the instance of the sequent calculus inference rule:

\[
\frac{\vdash a, \bar{a} \quad \vdash b, \bar{b}}{\vdash a, b, \bar{a} \land \bar{b}} \quad \text{R}^\land \\
\vdash a, b, \bar{a} \land \bar{b} \quad \leadsto \quad a \lor b \lor (\bar{a} \land \bar{b})
\]
However...

Consider the instance of the sequent calculus inference rule:

\[ \vdash a, \bar{a} \quad \vdash b, \bar{b} \quad R\wedge \quad \vdash a, b, \bar{a} \land \bar{b} \quad \leadsto \quad a \lor b \lor (\bar{a} \land \bar{b}) \]

In the calculus of structures this rule is simulated by the switch rule:

\[
\begin{align*}
\text{s} & \quad (a \lor \bar{a}) \land (b \lor \bar{b}) \\
\text{s} & \quad a \lor (\bar{a} \land (b \lor \bar{b})) \\
\text{s} & \quad a \lor b \lor (\bar{a} \land \bar{b})
\end{align*}
\]
However...

Consider the instance of the sequent calculus inference rule:

\[
\begin{align*}
\vdash a, \bar{a} & \vdash b, \bar{b} \\
\hline
\vdash a, b, \bar{a} \land \bar{b} \\
\vdash a, b, \bar{a} \land \bar{b} \quad \leadsto 
\end{align*}
\]

\[
\vdash a \lor b \lor (\bar{a} \land \bar{b})
\]

In the calculus of structures this rule is simulated by the switch rule:

\[
\begin{align*}
\frac{(a \lor \bar{a}) \land (b \lor \bar{b})}{s} \\
\frac{a \lor (\bar{a} \land (b \lor \bar{b}))}{s} \\
\frac{a \lor b \lor (\bar{a} \land \bar{b})}{s}
\end{align*}
\]

Switch rule can be applied to \(a \lor b \lor (\bar{a} \land \bar{b})\) in 27 different ways, and
However...

Consider the instance of the sequent calculus inference rule:

\[
\frac{\vdash a, \bar{a} \quad \vdash b, \bar{b}}{\vdash a, b, \bar{a} \land \bar{b}} \quad R\land \\
\vdash a, b, \bar{a} \land \bar{b} \quad \leadsto \quad a \lor b \lor (\bar{a} \land \bar{b})
\]

In the calculus of structures this rule is simulated by the switch rule:

\[
\begin{align*}
&s \quad (a \lor \bar{a}) \land (b \lor \bar{b}) \\
&s \quad a \lor (\bar{a} \land (b \lor \bar{b})) \\
&s \quad a \lor b \lor (\bar{a} \land \bar{b})
\end{align*}
\]

Switch rule can be applied to \( a \lor b \lor (\bar{a} \land \bar{b}) \) in 27 different ways, and to \( a_1 \lor b_1 \lor (\bar{a}_1 \land \bar{b}_1 \land (a_2 \lor b_2 \lor (\bar{a}_2 \land \bar{b}_2))) \) in 69 different ways.
However...

Consider the instance of the sequent calculus inference rule:

\[
\begin{array}{c}
\vdash a, \bar{a} \quad \vdash b, \bar{b} \\
\hline
\vdash a, b, \bar{a} \land \bar{b}
\end{array}
\quad R\land
\quad \vdash a, b, \bar{a} \land \bar{b} \quad \leadsto \quad a \lor b \lor (\bar{a} \land \bar{b})
\]

In the calculus of structures this rule is simulated by the switch rule:

\[
\begin{array}{c}
\text{s} \quad (a \lor \bar{a}) \land (b \lor \bar{b}) \\
\hline
\text{s} \quad a \lor (\bar{a} \land (b \lor \bar{b}))
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{s} \quad a \lor b \lor (\bar{a} \land \bar{b})
\]

Switch rule can be applied to \(a \lor b \lor (\bar{a} \land \bar{b})\) in 27 different ways, and to \(a_1 \lor b_1 \lor (\bar{a}_1 \land \bar{b}_1) \land (a_2 \lor b_2 \lor (\bar{a}_2 \land \bar{b}_2)))\) in 69 different ways.

Deep inference causes redundant nondeterminism.
System BV

- System BV: [Guglielmi,99] smallest technically nontrivial system

  MLL + mix + mix0 + a non-commutative self-dual operator

resembling prefix operator of process algebra: $a.b.P$
System BV

- **System BV**: [Guglielmi, 99] smallest technically nontrivial system

  \[
  \text{MLL} + \text{mix} + \text{mix0} + \text{a non-commutative self-dual operator}
  \]

  resembling prefix operator of process algebra: \(a.b.P\)

- **BV structures**:

  
  \[
  S ::= \circ \mid a \mid [S, \ldots, S] \mid (S, \ldots, S) \mid \langle S; \ldots; S \rangle \mid \bar{S}
  \]

  \[
  \begin{array}{c}
  >0 \\
  >0 \\
  >0 \\
  \end{array}
  \]
System BV

- **System BV**: [Guglielmi, 99] smallest technically nontrivial system

  \[ \text{MLL } + \text{ mix } + \text{ mix0 } + \text{a non-commutative self-dual operator} \]

  resembling prefix operator of process algebra: \( a \cdot b \cdot P \)

- **BV structures**:

  \[
  S ::= \circ | a | [S, \ldots, S] | (S, \ldots, S) | \langle S; \ldots; S \rangle | \bar{S} \\
  \quad >0 \quad >0 \quad >0
  \]

  \[
  [(\bar{a}, \bar{b}), a, b] \quad \text{corresponds to} \quad ((a^\perp \otimes b^\perp) \otimes a \otimes b)
  \]
System BV

- **System BV**: [Guglielmi,99] smallest technically nontrivial system

  \[ \text{MLL} + \text{mix} + \text{mix0} + \text{a non-commutative self-dual operator} \]

  resembling prefix operator of process algebra: \( a \cdot b \cdot P \)

- **BV structures**:

  \[
  S ::= \circ | a | [S, \ldots, S] | (S, \ldots, S) | \langle S; \ldots; S \rangle | \bar{S}
  \]

  \[
  [ (\bar{a}, \bar{b}), a, b ] \quad \text{corresponds to} \quad ((a^\perp \otimes b^\perp) \otimes a \otimes b)
  \]

- Structures are considered **equivalent modulo** an equational theory.
Syntactic Equivalence of BV Structures

**Associativity**

\[ [R, [T, U]] = [[R, T], U] \]

\( (R, (T, U)) = ((R, T), U) \)

\( \langle R; \langle T; U \rangle \rangle = \langle \langle R; T \rangle; U \rangle \)
Syntactic Equivalence of BV Structures

**Associativity**

\[
[R, [T, U]] = [[R, T], U] \\
(R, (T, U)) = ((R, T), U) \\
\langle R; \langle T; U \rangle \rangle = \langle \langle R; T \rangle; U \rangle
\]

**Commutativity**

\[
[R, T] = [T, R] \\
(R, T) = (T, R)
\]
Syntactic Equivalence of BV Structures

**Associativity**

\[
[R, [T, U]] = [[R, T], U] \\
(R, (T, U)) = ((R, T), U) \\
\langle R; \langle T; U \rangle \rangle = \langle \langle R; T \rangle; U \rangle
\]

**Commutativity**

\[
[R, T] = [T, R] \\
(R, T) = (T, R)
\]

**Unit**

\[
[\circ, R] = R \\
(\circ, R) = R \\
\langle R; \circ \rangle = R \\
\langle \circ; R \rangle = R
\]
Syntactic Equivalence of BV Structures

**Associativity**

\[ [R, [T, U]] = [[R, T], U] \]

\((R, (T, U)) = ((R, T), U)\)

\(\langle R; \langle T; U \rangle \rangle = \langle \langle R; T \rangle; U \rangle\)

**Commutativity**

\[ [R, T] = [T, R] \]

\((R, T) = (T, R)\)

**Unit**

\[ [\circ, R] = R \]

\((\circ, R) = R\)

\(\langle R; \circ \rangle = R\)

\(\langle \circ; R \rangle = R\)

**Negation**

\[ [\bar{R}, T] = (\bar{\bar{R}}, \bar{T}) \]

\((\bar{R}, T) = [\bar{\bar{R}}, \bar{T}]\)

\(\langle \bar{R}; T \rangle = \langle \bar{\bar{R}}; \bar{T} \rangle\)

\(\bar{\circ} = \circ\)

\(\bar{\bar{R}} = R\)
System BV of the Calculus of Structures

\[
\begin{align*}
\text{ai} & \downarrow \quad \frac{S\{\circ\}}{S[a, \bar{a}]} \\
\text{s} & \quad \frac{S([R, U], T)}{S[(R, T), U]} \\
\text{q} & \downarrow \quad \frac{S\langle[R, U]; [T, V]\rangle}{S[\langle R; T \rangle, \langle U; V \rangle]} 
\end{align*}
\]
System BV of the Calculus of Structures

\[
\begin{align*}
\text{ai} & \Downarrow \frac{S\{\circ\}}{S[a, \bar{a}]} & s & \frac{S([R, U], T)}{S[(R, T), U]} & q & \frac{S\langle[R, U]; [T, V]\rangle}{S\langle(R; T), \langle U; V\rangle\rangle}
\end{align*}
\]

\[
\Downarrow \quad \Downarrow
\]

\[
\text{MLL} \quad \left\{ \begin{array}{c}
\text{ai} \Downarrow \frac{S\{1\}}{S[a, \bar{a}]} & s & \frac{S([R, U], T)}{S[(R, T), U]}
\end{array} \right. 
\]
All the systems in the calculus of structures follows this scheme.
Classical Logic in the Calculus of Structures

\[
\begin{align*}
S\{\circ\} & \quad S([R, U], T) & \quad S\langle[R, U]; [T, V]\rangle \\
S[a, \bar{a}] & \quad S[(R, T), U] & \quad S[(R; T), (U; V)]
\end{align*}
\]
Classical Logic in the Calculus of Structures

\[ \text{ai} \downarrow \frac{S\{\circ\}}{S[a, \bar{a}]} \quad \text{s} \quad \frac{S([R, U], T)}{S[(R, T), U]} \quad \text{q} \downarrow \frac{S[[R, U]; [T, V]]}{S[\langle R; T \rangle, \langle U; V \rangle]} \]

\[ \text{KSg} \]

\[ \begin{align*}
\text{ai} \downarrow & \frac{S\{\texttt{t}\}}{S[a, \bar{a}]} \\
\text{w} \downarrow & \frac{S\{\texttt{f}\}}{S\{R\}} \\
\text{c} \downarrow & \frac{S[R, R]}{S\{R\}}
\end{align*} \]

[Brünnler, CSL’03]

[Brünnler, CSL’03]
Reducing Nondeterminism

In BV, the rule $s$ can be applied to $[(a, b), \bar{a}, \bar{b}]$ in 12 different ways:

\[
\begin{align*}
\frac{([\bar{a}, a, b], \bar{b})}{[(\bar{a}, \bar{b}), a, b]} & \quad \frac{([[\bar{a}, b], \bar{b}), a]}{[(\bar{a}, \bar{b}), a, b]} & \quad \frac{([\bar{a}, \bar{b}, a], b]}{[(\bar{a}, \bar{b}), a, b]}
\end{align*}
\]
Reducing Nondeterminism

In BV, the rule $s$ can be applied to $[(a, b), \bar{a}, \bar{b}]$ in 12 different ways:

\[
\begin{align*}
&s \frac{([\bar{a}, a, b], \bar{b})}{[(\bar{a}, \bar{b}), a, b]} \quad s \frac{([\bar{a}, b], \bar{b}), a]}{[(\bar{a}, \bar{b}), a, b]} \quad s \frac{([\bar{a}, \bar{b}, a], b]}{[(\bar{a}, \bar{b}), a, b]} \\
\end{align*}
\]

Observation: Switch rule breaks the “interaction” between atoms.

\[
\begin{align*}
&s \frac{S([R, W], T)}{S[(R, T), W]} \\
\end{align*}
\]
Reducing Nondeterminism

In BV, the rule $s$ can be applied to $[(a, b), \bar{a}, \bar{b}]$ in 12 different ways:

\[
\begin{align*}
\frac{([\bar{a}, a, b], \bar{b})}{((\bar{a}, \bar{b}), a, b)} & \quad s \quad \frac{([\bar{a}, b], \bar{b}), a]}{((\bar{a}, \bar{b}), a, b]} & \quad s \quad \frac{([\bar{a}, b], a)]}{((\bar{a}, \bar{b}), a, b]} & \quad s \quad \frac{([\bar{a}, \bar{b}], a)]}{((\bar{a}, \bar{b}), a, b]} \\
\end{align*}
\]

Observation: Switch rule breaks the “interaction” between atoms.

\[
\frac{\text{lis} \quad S([R, W], T)}{S([R, T], W]} \quad \text{if} \quad \text{at} \bar{R} \cap \text{at} W \neq \emptyset
\]

Definition: System BVsl is the system \{ai\downarrow, lis, q\downarrow\}. 
Lazy Interaction Switch

Consider:

$$[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])])$$

$$S([\{R, W\}, T])$$

$$S((R, T), W)$$
Lazy Interaction Switch

Consider:

\[ [a, b, (\overline{a}, \overline{b}, [c, d, (\overline{c}, \overline{d}, [e, f, (\overline{e}, \overline{f})]]))] \]

\[
\begin{align*}
S([R, W], T) \\
\text{s} \quad \frac{S([R, W], T)}{S([R, T], W)}
\end{align*}
\]

- The rule \( s \) can be applied to this structure in \( 42 \) different ways.
  (In system KSw, in 111 different ways.)
Lazy Interaction Switch

Consider:

\[[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])]]\]

\[
lis \frac{S([R, W], T)}{S((R, T), W)} \quad \text{if} \quad \text{at } \bar{R} \cap \text{at } W \neq \emptyset
\]

- The rule \textit{lis} can be applied in 14 different ways.
Lazy Interaction Switch

Consider:

\[
[b, ([a, \bar{a}], \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})]))])]
\]

\[
[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})]))])]
\]

\[
\text{lis } \frac{S([R, W], T)}{S([R, T], W)} \quad \text{if} \quad \text{at } \bar{R} \cap \text{at } W \neq \emptyset
\]

- The rule \text{lis} can be applied in 14 different ways.

\[
\{a\} \cap \{a\} \neq \emptyset
\]
Lazy Interaction Switch

Consider:

\[
[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [f, ([e, \bar{e}], \bar{f}])]))])
\]

\[
[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])]))]
\]

\[\text{lins} \quad \frac{S([R, W], T)}{S((R, T), W)} \quad \text{if} \quad \text{at } \bar{R} \cap \text{at } W \neq \emptyset\]

The rule lins can be applied in 14 different ways.

\[
\{e\} \cap \{e\} \neq \emptyset
\]
Lazy Interaction Switch

Consider:

\[
[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [f, ([e, \bar{e}], \bar{f}])])])]
\]

\[
[b, a, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])]
\]

\[
\text{lis} \quad \frac{S([R, W], T)}{S([R, T], W)} \quad \text{if} \quad \text{at } \bar{R} \cap \text{at } W \neq \emptyset
\]

- The rule s can be applied to this structure in 42 different ways. (In system KSg, in 111 different ways.)
- The rule lis can be applied in 14 different ways.

\[
\{a\} \cap \{\bar{b}, c, \bar{c}, d, \bar{d}, e, \bar{e}, f, \bar{f}\} = \emptyset
\]
Reducing Nondeterminism

\[
\begin{align*}
\text{ai} & \downarrow \frac{[b, \bar{b}]}{([a, \bar{a}], \bar{b}), b)} & \frac{([a, \bar{a}], [b, \bar{b}])}{((\bar{a}, \bar{b}), a, b)} \\
\text{lis} & \frac{([a, \bar{a}], [b, \bar{b}])}{((\bar{a}, \bar{b}), a, b)} \\
\text{ai} & \downarrow \frac{[a, \bar{a}]}{a, (\bar{a}, [b, \bar{b}])} & \frac{([a, \bar{a}], [b, \bar{b}])}{((\bar{a}, \bar{b}), a, b)} \\
\text{lis} & \frac{([a, \bar{a}], [b, \bar{b}])}{((\bar{a}, \bar{b}), a, b)} \\
\end{align*}
\]
Reducing Nondeterminism

\[
\begin{align*}
\text{ai} \downarrow \frac{[b, b]}{[\([a, \bar{a}], b\), b]} \quad & \text{lis} \quad \frac{([a, \bar{a}], [b, b])}{([a, \bar{a}], b), b} \\
\text{ai} \downarrow \frac{[a, \bar{a}]}{[a, (\bar{a}, [b, b])]}) \quad & \text{lis} \quad \frac{([a, \bar{a}], [b, b])}{[a, (\bar{a}, [b, b])]} \\
\text{ai} \downarrow \frac{[b, b]}{[\([a, \bar{a}], b\), b]} \quad & \text{lis} \quad \frac{([a, \bar{a}], [b, b])}{([a, \bar{a}], b), b} \\
\text{ai} \downarrow \frac{[a, \bar{a}]}{[a, (\bar{a}, [b, b])]}) \quad & \text{lis} \quad \frac{([a, \bar{a}], [b, b])}{[a, (\bar{a}, [b, b])]} \\
\text{ai} \downarrow \frac{[b, b]}{[\([a, \bar{a}], b\), b]} \quad & \text{lis} \quad \frac{([a, \bar{a}], [b, b])}{([a, \bar{a}], b), b} \\
\end{align*}
\]

In system \{s, ai\downarrow\} in the proof search space of \([([\bar{a}, b], a, b]\), there are 358 derivations including these 6 proofs, and no other proofs.
Reducing Nondeterminism

**Definition:** System BVsl is the system \( \{ a_i \downarrow, l_i \downarrow, q \downarrow \} \).

**Theorem:** Systems \( \{ a_i \downarrow, s, q \downarrow \} \) (BV) and BVsl are equivalent. [LPAR’06]
Reducing Nondeterminism

**Definition:** System BVsl is the system \{ai↓, lis, q↓\}.

**Theorem:** Systems \{ai↓, s, q↓\} (BV) and BVsl are equivalent. [LPAR’06]

**Corollary:** Systems \{ai↓, s\} and \{ai↓, lis\} are equivalent. [LPAR’06]
Definition: System BVsl is the system \{ai\downarrow, lis, q\downarrow\}.

Theorem: Systems \{ai\downarrow, s, q\downarrow\} (BV) and BVsl are equivalent. [LPAR’06]

Corollary: Systems \{ai\downarrow, s\} and \{ai\downarrow, lis\} are equivalent. [LPAR’06]

Theorem: The cut rule is admissible for system BVsl. [Tech.Rep.06]
Reducing Nondeterminism

**Definition:** System BVsl is the system \{ai↓, lis, q↓\}.

**Theorem:** Systems \{ai↓, s, q↓\} (BV) and BVsl are equivalent. [LPAR’06]

**Corollary:** Systems \{ai↓, s\} and \{ai↓, lis\} are equivalent. [LPAR’06]

**Theorem:** The cut rule is admissible for system BVsl. [Tech.Rep.06]

**Theorem:** System BV is NP-Complete. [WOLLIC’06]
Classical Logic in the Calculus of Structures

\[
\begin{align*}
\text{ai} & \quad \frac{S\{\circ\}}{S[a, \bar{a}]} \quad s \quad \frac{S([R, U], T)}{S[(R, T), U]} \quad q \quad \frac{S\langle[R, U]; [T, V]\rangle}{S[\langle R; T\rangle, \langle U; V\rangle]} \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
\text{KSg} & \quad \left\{
\begin{aligned}
\text{ai} & \quad \frac{S\{\top\}}{S[a, \bar{a}]} \quad s \quad \frac{S([R, U], T)}{S[(R, T), U]} \\
\text{w} & \quad \frac{S\{f\}}{S\{R\}} \quad c \quad \frac{S[R, R]}{S\{R\}}
\end{aligned}
\right. \\
\end{align*}
\]

[Brünnler, CSL’03]
Theorem: A structure \( R \) has a proof in KSg iff

\[
\begin{align*}
\not\vdash \{s, ai \downarrow\} & \quad R'' \\
\vdash \{w \downarrow\} & \quad [tt, tt] = tt \\
\vdash \{c \downarrow\} & \quad (ff, ff) = ff \\
\vdash \{\} & \quad R
\end{align*}
\]
Reducing Nondeterminism in Classical Logic System KSc

**Theorem:** A structure $R$ has a proof in KSc iff

$$\vdash \{s, ai\downarrow\}$$

$R''$

$\vdash \{w\downarrow\}$

$[tt, tt] = tt$

$R'$

$\vdash \{c\downarrow\}$

$(ff, ff) = ff$

$R$

**Definition:** System KSgi is the system resulting from replacing the switch rule in system KSc with the lazy interaction switch rule.
Reducing Nondeterminism in Classical Logic System KSg

**Theorem:** A structure $R$ has a proof in KSg iff

$$\vdash \{ s, ai \downarrow \}$$

$R''$

$\vdash \{ w \downarrow \}$

$\llbracket \tt, \tt \rrbracket = \tt$

$R'$

$\vdash \{ c \downarrow \}$

$(\ff, \ff) = \ff$

$R$

**Definition:** System KSgi is the system resulting from replacing the switch rule in system KSg with the lazy interaction switch rule.

**Theorem:** Systems KSg and KSgi are equivalent. [LPAR'06]
Implementation in Maude

- Systems in the calculus of structures can be expressed as term rewriting systems modulo equational theories.
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- Systems of the calculus of structures can be easily implemented by resorting to the simple high level language of Maude. [ESSLLI’04,ISCIS’04]
Example: Maude Module for System BV

mod BV is
  sorts Atom Unit Structure .
  subsort Atom < Structure .
  subsort Unit < Structure .

  op o : -> Unit .
  op -_ : Atom -> Atom [ prec 50 ] .
  ops a b c d e : -> Atom .

  var R T U V : Structure . var A : Atom .

  rl [q-down] : [ < R ; T > , < U ; V > ] => < [R,U] ; [T,V] > .
endm
Automated Proof Search

1. \([a, b, (\bar{a}, \bar{b}, [a, b, (\bar{a}, \bar{b})])]]\)

2. \([a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])]]\)

<table>
<thead>
<tr>
<th>Query</th>
<th>System</th>
<th># states explored</th>
<th>finds a proof in # ms (cpu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>{s, ai↓}</td>
<td>1041</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>{lis, ai↓}</td>
<td>264</td>
<td>0</td>
</tr>
<tr>
<td>2.</td>
<td>{s, ai↓}</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>{lis, ai↓}</td>
<td>6595</td>
<td>1370</td>
</tr>
</tbody>
</table>
Lazy Interaction Switch Revisited

Consider:

$$[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [f, ([e, \bar{e}], \bar{f}])]))])$$

$$[b, a, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])]))])$$

The rule $s$ can be applied to this structure in 42 different ways. (In system KSG, in 111 different ways.)

The rule $lis$ can be applied in 14 different ways.

$$\{a\} \cap \{\bar{b}, c, \bar{c}, d, \bar{d}, e, \bar{e}, f, \bar{f}\} = \emptyset$$
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\]

\[
\text{lis} \frac{S([R, W], T)}{S[(R, T), W]} \quad \text{if} \quad \text{at} \ R \cap \text{at} \ W \neq \emptyset
\]

- The rule \( s \) can be applied to this structure in 42 different ways. (In system KSG, in 111 different ways.)
- The rule \( \text{lis} \) can be applied in 14 different ways.

\[
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The condition of the rule must be performed for 42 such substructures. This is expensive in proof search.
Deep Inference vs. Deepest Inference

Idea: When we restrict the application of the inference rules to the deepest redexes, we are restricted to the smaller substructures.
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Is there a plausible notion of ”deepest inference” that is complete?

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**Definition:** A instance of the switch rule

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**Proposition:** Every proof in system \(\{ai\downarrow, s\}\) can be transformed to a proof in \(\{ai\downarrow, ds\}\) in linear time.
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- Providing a confluent deductive system for MLL for structures consisting of pairwise distinct atoms. [Guerrini, 1999]
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