A study of normalisation through subatomic logic

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Abstract

We introduce subatomic proof theory, a new methodology where, by looking *inside atoms*, we are able to represent a wide variety of proof systems in such a way that every rule is an instance of a single, regular, linear rule scheme. We show the generality of the subatomic approach by presenting how it can be applied to several different proof systems with very different expressivity.

In this thesis we use the subatomic approach to study two normalisation procedures: cut-elimination and decomposition. In particular, we study cut-elimination by characterising a whole class of substructural logics and giving a generalised cut-elimination procedure for them, and we study decomposition by providing generalised rewriting rules for derivations that we can then apply to decompose derivations.

Further, we exploit these rewriting rules to eliminate cycles and prove that cut-elimination and decomposition are independent from each other. We therefore obtain a modular normalisation theory, consisting of these two procedures.

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Contents

Proof theorists have long been interested in the study of normalisation of proofs. From cut-elimination to proof identity, finding a normal form for proofs is a valuable research goal that includes questions such as which properties we would like for the normal form, and what the size of the normal form is in relation to the original proof.

Proof normalisation also plays an important role in theoretical computer science, via the Curry-Howard-isomorphism [34], which identifies formulas and types. Proofs correspond to programs, and the normalization of the proof corresponds to the computation of the program. For example, the cut rule allows an auxiliary result to be proven only once, but used many times. When viewing proofs as programs, the cut is the application of a function to an argument, and normalisation is computation.

However, to study normalisation procedures with some generality is very difficult: cut-elimination procedures for example are highly sensitive to variations on the form and structure of the rules of a system, where a single change in one of the rules or the addition of another warrant the need for a full new proof of cut-elimination in a new system. In this thesis we unveil a common structure behind proof systems that will allow us to generalise and understand normalisation in a simpler and more effective way. We provide a new approach within the setting of deep inference, which we call *subatomic*. It allows us to present a wide variety of propositional proof systems in such a way that *every rule* is an instance of a *single* simple linear rule scheme. We exploit this generality to study normalisation procedures and their complexity, and in particular to unveil the role played by the interactions between the rules.

Gentzen's proofs of cut-elimination [15] for classical and intuitionistic logic were only the first instance of a type of argument that has been long studied since. From that breakthrough, Gentzen-style cut elimination proofs abound in the literature, exploring on a system-by-system basis how to permute the cut-rules towards the premiss of a proof. The specificity needed for these cut-elimination arguments requires tricky case by case analyses, making it difficult to understand *how* cut-elimination works. Indeed, when designing a new proof system a complex trial and error phase is necessary to obtain cut admissibility. The fact that simple variations of a rule have so much influence on these arguments is the first hint that cut-elimination is in fact a combinatorial phenomenon, hinging mostly on the shape and interaction between the rules of a system.

In particular, in traditional Gentzen-style cut-elimination procedures cut instances are eliminated from proofs by moving upwards instances of the *mix* rule [16, 14]:

$$\frac{\vdash mA, \Gamma \quad \vdash n\bar{A}, \Delta}{\vdash \Gamma, \Delta} \quad .$$

This rule conflates one instance of cut and several instances of contraction and therefore by using this technique we are in fact observing two different interactions between rules: the interactions of the cut with other rules, and the interactions of contractions with other rules. This phenomenon becomes more apparent when one considers the complexity of cut-elimination in different systems: in purely linear systems such as multiplicative linear logic the procedure does not change the size of proofs significantly, whereas as soon as contractions are introduced the size of proofs can grow exponentially or more.

In what follows we aim to move towards a generalised modular normalisation theory where the different interactions between rules are dealt with separately providing a tighter control over complexity creation. Furthermore, by separating normalisation into two independent procedures, we may provide some guidance towards the development of computational interpretations that may interpret each procedure as a particular kind of computation.

Since our main aim is to study the interactions between rules, we will do so in the setting of deep inference [26, 42] where rules can be reduced to their atomic form providing great regularity in the inference rule schemes and where atomic contractions can be permuted through cuts and confined to the bottom of a proof.

In deep inference proofs can be composed by the logical connectives that are used to compose formulae [28]. For example, if $\phi = \begin{array}{c} A \\ \| \\ B \end{array}$ and $\psi = \begin{array}{c} C \\ \| \\ D \end{array}$ are two proofs in propositional logic,

$$\phi \wedge \psi = \begin{array}{c} A \\ \| \\ B \end{array} \wedge \begin{array}{c} C \\ \| \\ D \end{array} \quad \text{and} \quad \phi \vee \psi = \begin{array}{c} A \\ \| \\ B \end{array} \vee \begin{array}{c} C \\ \| \\ D \end{array}$$

are two valid proofs with premisses $A \wedge C$ and $A \vee C$ and conclusions $B \wedge D$ and $B \vee D$ respectively. In deep inference, rules can be applied at any depth inside a formula and as a result every contraction and cut instance can be locally transformed into their atomic variants by a local procedure of polynomial-size complexity [6, 41, 5]. This provides a surprising regularity in the inference rule schemes: it can actually be observed that in most deep inference systems all rules besides the atomic ones can be expressed as

$$\frac{(A \;\alpha\; B) \;\beta\; (C \;\gamma\; D)}{(A \;\epsilon\; C) \;\zeta\; (B \;\eta\; D)} \quad ,$$

where A, B, C, D are formulae and $\alpha, \beta, \gamma, \epsilon, \zeta, \eta$ are logical relations. We call this rule shape a *medial shape*.

Following this observations, in this thesis we discover a way to present *every rule* as an instance of a rule with the medial shape. This characterisation is not trivial: it is a delicate trade-off to impose restrictions on the possible assignments for $\alpha, \beta, \gamma, \epsilon, \zeta, \eta$ that allow us to characterise systems that enjoy cut-elimination, but that are general enough to encompass a wide variety of logics. Indeed, the finding of these restrictions is the product of a long trial-and-error phase to obtain the desired generality together with the desired properties.

The main idea of this work is to consider atoms as self-dual, noncommutative binary logical relations and to build formulae by freely composing units by atoms and the other logical relations. We will consider the occurrences of an atom a as interpretations of more primitive expressions involving a noncommutative binary relation, still denoted by a. Two formulae A and B in the relation a, in this order, are denoted by $A\,a\,B$. Formulae are built over the units for the logical relations, denoted for example by t, f in the case of classical logic. We can think of it as a superposition of truth values: $\mathsf{f}\,a\,\mathsf{t}$ is the superposition of the two possible assignments for the atom a. We can for example have a projection onto a specific assignment by choosing which 'side' we read: if we read the values on the left of the atom we assign f to a and if we read the ones on the right we assign t to a. We call these formulae *subatomic*. For example,

$$((\mathsf{t}\,a\,\mathsf{f}) \wedge (\mathsf{f}\,b\,\mathsf{f})) \vee ((\mathsf{f} \wedge \mathsf{t})\,a\,\mathsf{t}) \quad \text{and} \quad (\mathsf{t}\,a\,\mathsf{t})\,b\,(\mathsf{f} \wedge \mathsf{f})$$

are subatomic formulae for classical logic.

In this way, we obtain an extended language of formulae which we can relate to the usual propositional formulae, or *interpret*, through an interpretation map $\overset{I}{\mapsto}$. A natural way to build such a map is to provide meaning to units inside the scope of an atom, by setting $\mathsf{f}\,a\,\mathsf{t} \overset{I}{\mapsto} a$ and $\mathsf{t}\,a\,\mathsf{f} \overset{I}{\mapsto} \bar{a}$, and extending it to all formulae in the natural way.

Subatomic formulae are much more than a clever representation. By using them, we are strikingly able to present proof systems in such a way that *every rule* has a medial shape, *including the atomic rules* that do not usually follow this scheme. For example, the rules for atomic introduction and atomic contraction can be represented as

$$\frac{(\mathsf{f} \vee \mathsf{t})\,a\,(\mathsf{t} \vee \mathsf{f})}{(\mathsf{f}\,a\,\mathsf{t}) \vee (\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \frac{\mathsf{t}}{a \vee \bar{a}} \quad \text{and} \quad \frac{(\mathsf{f}\,a\,\mathsf{t}) \vee (\mathsf{f}\,a\,\mathsf{t})}{(\mathsf{f} \vee \mathsf{f})\,a\,(\mathsf{t} \vee \mathsf{t})} \overset{I}{\mapsto} \frac{a \vee a}{a} \quad .$$

This provides us with a useful way to reason generally about proof systems: we need only focus on how rules with this shape interact with each other.

There are many different cut-elimination techniques in the deep inference literature [21, 4, 3, 41, 31], exploiting different aspects of the proof systems they work on. In this assortment, a particular methodology does however stand out for its generality: cut-elimination via *splitting* [26] can be achieved in the deep inference systems for linear logic [39], multiplicative exponential linear logic [41], the mixed commutative/non-commutative logic BV [26] and its extension with linear exponentials NEL [31], and classical predicate logic [4]. The generality of this procedure points towards the fact that it exploits some properties that are common to all these systems.

Splitting is based on a simple idea: to show that an atomic cut involving a and $\bar{a}$ is admissible, we trace a and $\bar{a}$ to the top of the proof to find two independent subproofs, the premiss of one containing the dual of a and the other one containing the dual of $\bar{a}$. In this way we obtain two independent 'pieces' that we can rearrange to get a new cut-free proof.

$$\dfrac{\begin{array}{c} H_a \otimes \dfrac{1}{\bar a \parr a} \\ \| \\ K_a \parr a \end{array} \otimes \begin{array}{c} \dfrac{1}{\bar a \parr a} \otimes H_{\bar a} \\ \| \\ \bar a \parr K_{\bar a} \end{array}}{K_a \parr \dfrac{a \otimes \bar a}{\bot} \parr K_{\bar a}} \quad \xrightarrow{\text{splitting}} \quad \dfrac{H_a \otimes \dfrac{1}{\bar a \parr a} \otimes H_{\bar a}}{\boxed{\begin{array}{c} H_a \otimes \bar a \\ \| \\ K_a \end{array}} \parr \boxed{\begin{array}{c} a \otimes H_{\bar a} \\ \| \\ K_{\bar a} \end{array}}}$$

This type of argument has been used to prove the admissibility of rules other than the atomic cut [26], showing that it can be applied to any logical relation that we can trace upwards in a proof just like we traced the atoms in the above argument. Thus, the splitting procedure hinges strongly on the dualities present in propositional logical systems (to find the duals of a and $\bar{a}$) and on the regularity of deep inference rules (to follow the atoms in a proof), further confirming the suspicion that logical dualities and the shape of rules have a strong bearing on cut-elimination.

Based on this intuition, we capitalise on the regularity of subatomic inference rules to generalise this process, studying which rules allow us to follow a connective in a proof. We show that in systems where the scope of relations only increases reading from bottom to top, called *splittable* systems, we can follow these relations through the proof and hence a whole class of rules is admissible via the splitting procedure. Splittable systems turn out to be the subatomic equivalent to propositional systems that we would characterise as linear, i.e., having no contractions. Unsurprisingly then, the class of rules shown admissible is precisely the class of rules that allow us to make the cut atomic in deep inference formalisms.

Achieving this simple characterisation of splittable systems gives us a full understanding of *how* the splitting procedure works, and *why* it has been used with success to prove the admissibility of different rules in several systems. We note that splitting is a global procedure: we need to study the proof as a whole to obtain a cut-free proof through splitting. Furthermore, splitting does not create meaningful complexity: the size of the cut-free proofs obtained by general splitting is linear on the size of the proofs with cut they come from, and splitting is a procedure of polynomial-time complexity. This is an interesting observation for the further study of complexity, since deep inference proofs are at most as big as sequent proofs [8].

Splitting allows us to understand the interactions of the cut with linear rules, but how about contractions? It is known that we can decompose a classical logic proof into a linear phase and a phase made-up only of contractions [32], or that we can decompose a first-order proof into a propositional phase and a quantified phase through a Herbrand theorem [9, 4]. These type of results are called decomposition theorems, and they provide normal forms for proofs that are of great use since they allow us to separate proofs into different fragments that we can study independently from each other.

We will study this phenomenon, providing general rewriting rules that correspond to the rewritings presented both for classical logic and for MALL in [32] and [39] where atomic contractions can be confined to the bottom of proofs. We will thus show that both decomposition results are a consequence of precisely the same properties.

Additionally, it has long been conjectured [6] that it is possible to achieve a further decomposition of these systems, permuting not only the atomic contraction but a whole family of *contractive* rules towards the bottom of a derivation. The generalised rewriting rules that we present allow us to permute contractive rules with linear rules, including cuts. The regularity provided by subatomic systems is a big simplification for the study of these interactions: by having a single shape we only have to consider two non-trivial permutation cases.

Lastly, decomposition for classical logic has been proved to be independent from cut-elimination only in the case of cycle-free proofs [32]. Cycles are a particular construction that might occur in a proof with cuts and contractions, and it is known that it is possible to remove them as a consequence of cut-elimination. Cycles have been studied in the sequent calculus, and it has been shown that removing them might entail an exponential complexity growth [12]. Through our generalised rewriting rules we are able to present a purely local procedure based on permutations to remove the cycles in proofs, fully showing that decomposition in classical logic is independent from cut-elimination. Furthermore, this procedure will allow us to be able to study the complexity cost of the elimination of cycles in deep inference independently from cut-elimination, which is as of now unknown.

In this thesis we present the following results:

- We formalise subatomic logic and show how it encompasses such different systems as multiplicative additive linear logic, BV and classical logic. We exploit its uniformity to study the effect of the interactions between rules in normalisation procedures.

- We present a generalisation of the splitting procedure, together with simple sufficient conditions for a system to enjoy splitting, that can be applied to a whole class of substructural logics to prove the admissibility of a family of cut-like rules, including the atomic cut. Logics that verify the conditions include multiplicative linear logic, the linear fragment of classical logic, and BV. Furthermore, we show that the splitting procedure is not restricted to systems with binary connectives and can be further generalised to relations of different arities by extending the splitting theorem to SKV, a system with a modality.

In addition, this generalisation provides useful guidelines for the design of linear proof systems, removing the search for cut-elimination from the design process.

- We provide a generalisation of decomposition reduction rules, together with sufficient conditions for a system to be decomposable into phases containing only atomic contractions/cocontractions and a linear phase. In this way we show that this type of decomposition result holds for example for both classical logic and multiplicative additive linear logic because of shared properties in the shape of their rules.
- We use the general reduction rules introduced in this thesis to design a local procedure to remove cycles, effectively proving the independence of decomposition and cut-elimination. This procedure can be applied to both classical logic and multiplicative additive linear logic.

In other words, we provide a new methodology that proves itself to be useful in its generality, allowing us to generalise and understand normalisation procedures in such a way that they capture several differently expressive logics. For this reason, this research aims to be only the start of the characterisation of proof systems and their properties by the shape of their rules, as well as a useful reference for proof system design.

Chapter 1

Subatomic Logic

In this chapter, we will show how to achieve complete regularity on the shape of inference rules by introducing a new methodology, that we call *subatomic* because we look 'inside the atoms'. We will start by introducing subatomic formulae and giving tools to relate them to 'ordinary formulae'. Subatomic formulae are built by freely composing constants by connectives and atoms. For example,

$$A \equiv ((\mathsf{f}\; a\; \mathsf{t}) \vee \mathsf{t}) \wedge (\mathsf{t}\; b\; \mathsf{f}) \quad \text{and} \quad B \equiv ((\mathsf{t}\; b\; \mathsf{f}) \wedge \mathsf{t}) \vee \mathsf{f}$$

are two subatomic formulae for classical logic. The main idea is to interpret $\mathsf{f}\; a\; \mathsf{t}$ as a positive occurrence of the atom a, and $\mathsf{t}\; a\; \mathsf{f}$ as a negative occurrence of the same atom, denoted by $\bar{a}$. Intuitively, we can view subatomic formulae as a superposition of truth values. For example, $\mathsf{f}\; a\; \mathsf{t}$ is the superposition of the two possible assignments for the atom a, and $\mathsf{t}\; a\; \mathsf{f}$ is the superposition of the possible assignments for $\bar{a}$: if we read the value on the left of the atom we assign f to a and t to $\bar{a}$, and if we read the one on the right we assign t to a and f to $\bar{a}$.

Since we consider atoms as connectives, we will give a broad definition of what relations are, not assuming any logical characteristics or properties such as commutativity or associativity. We will therefore encompass logics with both commutative and non-commutative, associative and non-associative, dual and-self dual relations. This feature deserves to be highlighted since expressing self-dual non-commutative connectives into proof systems that enjoy cut-elimination is a challenge in Gentzen-style sequent calculi: it is impossible to have a complete analytic system with a self-dual non-commutative relation [42].

Using the new structure offered by subatomic formulae together with the regularity provided by deep inference we will then show that it is possible to achieve full regularity on the shape of inference rules in a wide variety of systems. In deep inference, the possibilty of composing proofs with the same connectives as formulae allows us to reduce most rules to their atomic form. The inference rules so obtained present a surprising regularity, that we can exploit towards obtaining a general rule scheme that encompasses every inference rule. We will show an underlying structure on the shape of the inference rules, using it to present all the rules of a system as instances of a single rule scheme, including the atomic ones.

Consider for example system SKS for classical logic [6].

$$
ai\!\downarrow \frac{\mathsf{t}}{a \vee \bar{a}} \qquad ai\!\uparrow \frac{a \wedge \bar{a}}{\mathsf{f}}
$$

$$
s \frac{(A \vee B) \wedge C}{(A \wedge C) \vee B} \qquad m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}
$$

$$
ac\!\downarrow \frac{a \vee a}{a} \qquad ac\!\uparrow \frac{a}{a \wedge a}
$$

$$
aw\!\downarrow \frac{\mathsf{f}}{a} \qquad aw\!\uparrow \frac{a}{\mathsf{t}}
$$

System SKS

We can derive the rule s from the rule

$$
\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \quad ,
$$

which has the same 'shape' as the rule m. In fact we will show that in many systems most non-atomic rules can be made to fit this scheme as well. By using the subatomic methodology, we are able to further extend this phenomenon to atomic rules in such a way that we can present a system for classical logic where every rule of the system has the same shape.

$$
a\!\downarrow \frac{(A \vee B)\, a\, (C \vee D)}{(A\, a\, C) \vee (B\, a\, D)} \qquad a\!\uparrow \frac{(A\, a\, B) \wedge (C\, a\, D)}{(A \wedge C)\, a\, (B \wedge D)}
$$

$$
\wedge\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \vee\!\uparrow \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}
$$

$$
m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}
$$

$$
ac \frac{(A\, a\, B) \vee (C\, a\, D)}{(A \vee C)\, a\, (B \vee D)} \qquad a\bar{c} \frac{(A \wedge B)\, a\, (C \wedge D)}{(A\, a\, C) \wedge (B\, a\, D)}
$$

System SAKS

We will present a characterisation of this rule shape, showcasing its generality by presenting examples of several such regular systems for different logics, which will be extended with further examples throughout the rest of the thesis.

Lastly, we will extend the notion of proof to subatomic systems, in order to relate them to 'usual' proof systems.

1.1 Subatomic formulae

Subatomic formulae are built by freely composing constants by connectives and atoms. For example, $A \equiv ((\mathsf{f}\, a\, \mathsf{t}) \vee \mathsf{t}) \wedge (\mathsf{t}\, b\, \mathsf{f})$ and $B \equiv ((\mathsf{t}\, b\, \mathsf{f}) \wedge \mathsf{t}) \vee \mathsf{f}$ are two subatomic formulae for Classical Logic. By considering atoms as relations we will work with an extended language of formulae, since we can have atoms in the scope of other atoms, something that does not occur in 'traditional' formulae.

Definition 1.1. Let $\mathscr{U}$ be a denumerable set of *constants* whose elements are denoted by $u, v, w, \ldots$. Let $\mathscr{R}$ be a denumerable partially ordered set of *relations* whose elements are denoted by α, β, γ, ... The set $\mathscr{F}$ of *subatomic formulae* (or SA formulae) contains terms defined by the grammar

$$\mathscr{F} ::= \mathscr{U} \mid \mathscr{F}\ \mathscr{R}\ \mathscr{F} \quad .$$

Formulae are denoted by A, B, C, ...

A *(formula) context* $K\{\ \}\cdots\{\ \}$ is a formula where some subformulae are substituted by holes; $K\{A_1\}\cdots\{A_n\}$ denotes a formula where the n holes in $K\{\ \}\cdots\{\ \}$ have been filled with $A_1, \ldots, A_n$.

The expression $A \equiv B$ means that the formulae A and B are syntactically equal. We omit parentheses when there is no ambiguity.

In $K\{A\ \alpha\ B\}$ we say that the subformulae of A and B are *in the scope of* α.

Example 1.2. The set $\mathscr{F}_{cl}$ of subatomic formulae for classical logic is given by the set of constants $\mathscr{U} = \{\mathsf{f}, \mathsf{t}\}$ and the set of relations $\mathscr{R} = \{\wedge, \vee\} \cup \mathscr{A}$ where $\mathscr{A}$ is a denumerable set of atoms, denoted by $a, b, \ldots$ with $\mathscr{A} \cap \{\wedge, \vee\} = \varnothing$. Two examples of subatomic formulae for classical logic are

$$A \equiv ((\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{t})) \wedge (\mathsf{t}\ b\ \mathsf{f}) \quad \text{and} \quad B \equiv ((\mathsf{t}\ b\ \mathsf{f}) \wedge \mathsf{t}) \vee (\mathsf{f}\ a\ \mathsf{f}) \quad .$$

Example 1.3. The set $\mathscr{F}_{ll}$ of subatomic formulae for multiplicative linear logic is given by the set of constants $\mathscr{U} = \{\perp, 1\}$ and the set of relations $\mathscr{R} = \{⅋, \otimes\} \cup \mathscr{A}$ where $\mathscr{A}$ is a denumerable set of atoms, denoted by $a, b, \ldots$ with $\mathscr{A} \cap \{⅋, \otimes\} = \varnothing$. Two examples of subatomic formulae for linear logic are

$$C \equiv ((1 ⅋ \perp)\ a\ 1) \otimes \perp \quad \text{and} \quad D \equiv ((\perp ⅋ 1)\ b\ 1) \otimes (1\ a\ \perp) \quad .$$

Aside from classical logic and multiplicative linear logic, we will feature the logic BV [26] amongst the examples to showcase a well-studied logic with self-dual non-commutative connectives. For that, we define the logic BVU. BV will correspond to BVU with all the units identified.

Example 1.4. We define system BVU. The formulae of BVU are built from the units $\perp, \circ, 1$ by composing them with the relations $⅋, \triangleleft, \otimes$.

The relations $⅋$ and $\otimes$ are dual to each other, associative, commutative and have units $\bot$ and 1 respectively. $\triangleleft$ is self-dual and associative, and has unit $\circ$.

Negation on BVU formulae is built respecting DeMorgan dualities, with $\bar{\circ} = \circ$, and $\bar{\bot} = 1$.

The units verify the equations $\circ ⅋ \circ = 1$; $\circ \otimes \circ = \bot$ and $1 \triangleleft 1 = 1$; $\bot \triangleleft \bot = \bot$.

The inference rules for system BVU are given by the same rules as for system BV [26]. System BV corresponds to system BV with the three units identified, i.e. $1 = \circ = \bot$.

The set $\mathscr{F}_{bv}$ of subatomic formulae for the non-commutative logics BVU and BV is given by the set of constants $\mathscr{U} = \{\bot, 1, \circ\}$ and the set of relations $\mathscr{R} = \{⅋, \triangleleft, \otimes\} \cup \mathscr{A}$ where $\mathscr{A}$ is a denumerable set of atoms, denoted by $a, b, \ldots$ with $\mathscr{A} \cap \{⅋, \triangleleft, \otimes\} = \varnothing$. Two examples of subatomic formulae for BV are

$$E \equiv (1\ a\ \bot) \triangleleft (\circ \otimes (\bot\ b\ \bot)) \quad \text{and} \quad F \equiv ((\circ \otimes 1)\ a\ 1) ⅋ 1 \quad .$$

Just like for 'ordinary' formulae, we will define an equational theory and a negation map on the set of subatomic formulae. We will work in a classical setting, in the sense that we will consider an involutive negation that satisfies DeMorgan dualities. Furthermore, in order to keep track of the equational theory in the general results exposed in this thesis, we restrict the equalities that we allow.

Definition 1.5. We define *negation* as a pair of involutive maps $\bar{\cdot} : \mathscr{R} \mapsto \mathscr{R}$ and $\bar{\cdot} : \mathscr{U} \mapsto \mathscr{U}$. We define the *negation map on formulae* as the map inductively defined by setting $\overline{A\ \alpha\ B} := \overline{A}\ \overline{\alpha}\ \overline{B}$.

We define an equational theory $=$ on $\mathscr{F}$ as the minimal equivalence relation closed under negation (if $A = B$, then $\bar{A} = \bar{B}$) and under context (if $A = B$, then $K\{A\} = K\{B\}$ for any context $K\{\ \}$) defined by a set of axioms of the form:

(1) $\forall A, B, C \in \mathscr{F}.\ (A\ \alpha\ B)\ \alpha\ C = A\ \alpha\ (B\ \alpha\ C)$; *(Associativity of α)*

(2) $\forall A, B \in \mathscr{F}.\ A\ \alpha\ B = B\ \alpha\ A$; *(Commutativity of α)*

(3) $\forall A \in \mathscr{F}.\ A\ \alpha\ u_\alpha = A = u_\alpha\ \alpha\ A$ for a fixed $u_\alpha \in \mathscr{U}$; *(Unit of α)*

(4) $v\ \alpha\ w = u$ for fixed $v, w, u \in \mathscr{U}$; *(Constant assignment for α)*

(5) $u = v$ for fixed $u, v \in \mathscr{U}$. *(Constant identification)*

If there is an axiom of the form (1) for α, we say that α is *associative*. If there is an axiom of the form (2) for α, we say that α is *commutative*. If there is an axiom of the form (3) for α we say that α is *unitary*, and we call u_α the *unit of* α.

Remark 1.6. Since the equational theory is closed under negation, if α is unitary with unit u_α, then $\overline{\alpha}$ is unitary and its unit is $\overline{u_\alpha}$.

Example 1.7. For the set of subatomic formulae for classical logic $\mathscr{F}_{cl}$ defined in example 1.2, we define negation through:

$$\begin{aligned}
&\bar{\wedge} := \vee \ ;\\
&\bar{a} := a \text{ for all } a \in \mathscr{A} \ ;\\
&\bar{\mathsf{t}} := \mathsf{f} \ .
\end{aligned}$$

We define the equational theory $=$ on $\mathscr{F}_{cl}$ as the minimal equivalence relation closed under negation and under context defined by:

For all $A, B, C \in \mathscr{F}$:

$$\begin{aligned}
&(A \wedge B) \wedge C = A \wedge (B \wedge C) \ ; && (A \vee B) \vee C = A \wedge (B \vee C) \ ;\\
&A \wedge B = B \wedge A \ ; && A \vee B = B \vee A \ ;\\
&A \wedge \mathsf{t} = A \ ; && A \vee \mathsf{f} = A \ ;\\
&\mathsf{f} \wedge \mathsf{f} = \mathsf{f} \ ; && \mathsf{t} \vee \mathsf{t} = \mathsf{t} \ ;\\
&\forall a \in \mathscr{A}.\ \mathsf{f}\ a\ \mathsf{f} = \mathsf{f} \ ; && \forall a \in \mathscr{A}.\ \mathsf{t}\ a\ \mathsf{t} = \mathsf{t} \ .
\end{aligned}$$

Example 1.8. For the set of subatomic formulae for linear logic $\mathscr{F}_{ll}$ defined in example 1.3, we define negation through:

$$\begin{aligned}
&\bar{\otimes} = \parr \ ;\\
&\bar{a} := a \text{ for all } a \in \mathscr{A} \ ;\\
&\bar{1} := \bot \ .
\end{aligned}$$

We define the equational theory $=$ on $\mathscr{F}_{ll}$ as the minimal equivalence relation closed under negation and under context defined by:

For all $A, B, C \in \mathscr{F}$:

$$\begin{aligned}
&(A \otimes B) \otimes C = A \otimes (B \otimes C) \ ; && (A \parr B) \parr C = A \parr (B \parr C) \ ;\\
&A \otimes B = B \otimes A \ ; && A \parr B = B \parr A \ ;\\
&A \otimes 1 = A \ ; && A \parr \bot = A \ ;\\
&\forall a \in \mathscr{A}.\ \bot\ a\ \bot = \bot \ ; && \forall a \in \mathscr{A}.\ 1\ a\ 1 = 1 \ .
\end{aligned}$$

Example 1.9. For both BVU and BV we will define the same negation map. They will differ only on the equational theory, since all the units are identified in BV.

For the set of subatomic formulae for BVU and for BV $\mathscr{F}_{bv}$ defined in example 1.4, we define negation through:

$$
\begin{aligned}
&\bar{\otimes} := \parr\ ;\\
&\bar{\triangleleft} := \triangleleft\ ;\\
&\bar{a} := a \text{ for all } a \in \mathscr{A}\ ;\\
&\bar{\circ} := \circ\ ;\\
&\bar{\bot} := \mathsf{1}\ .
\end{aligned}
$$

For the logic BVU we define an equational theory $=$ on $\mathscr{F}_{bv}$ as the minimal equivalence relation closed under negation and under context defined by:

For all $A, B, C \in \mathscr{F}$:

$$
\begin{aligned}
&(A \otimes B) \otimes C = A \otimes (B \otimes C)\ ; && (A \parr B) \parr C = A \parr (B \parr C)\ ;\\
&A \otimes B = B \otimes A\ ; && A \parr B = B \parr A\ ;\\
&(A \triangleleft B) \triangleleft C = A \triangleleft (B \triangleleft C)\ ; && \\
&A \otimes \mathsf{1} = A\ ; && A \parr \bot = A\ ;\\
&A \triangleleft \circ = A\ ; && \circ \triangleleft A = A\ ;\\
&\circ \otimes \circ = \bot\ ; && \circ \parr \circ = \mathsf{1}\ ;\\
&\forall a \in \mathscr{A}.\ \bot\, a\, \bot = \bot\ ; && \forall a \in \mathscr{A}.\ \mathsf{1}\, a\, \mathsf{1} = \mathsf{1}\ ;\\
&\bot \triangleleft \bot = \bot\ ; && \mathsf{1} \triangleleft \mathsf{1} = \mathsf{1}\ .
\end{aligned}
$$

The equational theory for the logic BV defined on the set of subatomic formulae $\mathscr{F}_{bv}$ is given by the previous equations, together with the added axioms:

$$
\mathsf{1} = \circ\ ; \qquad\qquad \bot = \circ\ .
$$

Given a propositional logic with certain relations and constants, its subatomic counterpart is therefore composed of an extended language of formulae, made up from the same relations but with the added possibility of having atoms in the scope of other atoms. To translate the subatomic formulae into the 'usual' formulae, we can define a simple interpretation map.

The intuitive idea behind the translation is to interpret a certain assignment of units inside an atom as a positive occurrence of the atom, and the dual assignment as a negative occurrence of the atom. For example, for classical logic we interpret $\mathsf{f}\, a\, \mathsf{t}$ as a positive occurrence of the atom a and $\mathsf{t}\, a\, \mathsf{f}$ as a negative one. In this way, the formula $A \equiv ((\mathsf{f}\, a\, \mathsf{t}) \vee \mathsf{t}) \wedge (\mathsf{t}\, b\, \mathsf{f})$ is interpreted as $A' \equiv (a \vee \mathsf{t}) \wedge \bar{b}$.

We can view the constants in the scope of an atom as a superposition of truth values. $\mathsf{f}\, a\, \mathsf{t}$ is the superposition of the two possible assignments for the atom a and $\mathsf{t}\, a\, \mathsf{f}$ the superposition of the two assignments for $\bar{a}$. We can project onto a specific assignment by choosing which 'side' we read: if we read the values on the left of the atom we assign f to a and t to $\bar{a}$ and if we read the ones on the right we assign t to a and f to $\bar{a}$.

In order to define an interpretation map following this idea, subatomic formulae must be built from the same relations as the 'original' formulae, with the addition of the atoms as connectives.

Definition 1.10. Let $\mathcal{G}$ be the set of formulae of a propositional logic L. We say that the set of subatomic formulae $\mathcal{F}$ is *natural* for L if there is a partition on the set of relations $\mathcal{R} = \mathcal{A} \cup \mathcal{R}'$ with $\mathcal{A} \cap \mathcal{R}' = \varnothing$,such that:

- there is an injective map from the constants of $\mathcal{G}$ to the constants in $\mathcal{U}$;
- there is a one to one correspondence between the relations in $\mathcal{G}$ and the relations in $\mathcal{R}'$;
- there is a one to one correspondence between the set of unordered pairs of dual atoms $\{a, \bar{a}\}$ in $\mathcal{G}$ and the set of relations $\mathcal{A}$.

We call the relations in $\mathcal{A}$ *atoms* as well. For each distinct pair of dual atoms we give a polarity assignment: we call one atom of the pair *positive*, and the other one *negative*. We will denote the atom of $\mathcal{A}$ corresponding to each pair with the same letter as the positive atom of the pair.

We will denote the constants of $\mathcal{U}$ and the relations in $\mathcal{R}'$ with the same symbols as their counterparts in $\mathcal{G}$.

Example 1.11. The sets of subatomic formulae defined in examples 1.7, 1.8 and 1.9 are natural for classical logic, multiplicative linear logic and BV respectively.

The notion of interpretation map is easily extended to all logics for which we define a subatomic logic in the natural way. This interpretation will allow us to go back and forth between subatomic systems and 'regular' propositional systems.

Definition 1.12. Let $\mathcal{G}$ be the set of formulae of a propositional logic L with negation denoted by $\bar{\cdot}$ and equational theory denoted by $=$. Let $\mathcal{F}$ be the set of subatomic formulae with constants $\mathcal{U}$ and relations $\mathcal{R}$ with negation denoted by $\bar{\cdot}$ and equational theory denoted by $=$. A surjective partial function $I : \mathcal{F} \to \mathcal{G}$ is called *interpretation map*. The domain of definition of I is the *set of interpretable formulae* and is denoted by $\mathcal{F}^i$. If $F \equiv I(A)$, we say that F is the *interpretation* of A, and that A is a *representation* of F.

We extend the notion of interpretability to contexts: we say that $S\{\ \}$ is interpretable if $S\{A\}$ is interpretable for every interpretable A.

If $\mathcal{F}$ is natural for L, we say that an interpretation $i : \mathcal{F}^i \to \mathcal{G}$ is *natural* when:

- $I(u) \equiv u$ for every constant u of $\mathcal{G}$;
- $\forall \alpha \in \mathcal{R}'$, if $A, B \in \mathcal{F}^i$ then $A\ \alpha\ B \in \mathcal{F}^i$ and $I(A\ \alpha\ B) \equiv I(A)\ \alpha\ I(B)$;
- For some distinguished constants $u_1, u_2 \in \mathcal{U}$, for all $a \in \mathcal{A}$, $I(u_1\ a\ u_2) \equiv a$ and $I(u_2\ a\ u_1) \equiv \bar{a}$.

We define the *natural representation* $R : \mathcal{G} \to \mathcal{F}$ *associated to* I for every formula $G \in \mathcal{G}$ inductively on the structure of G by:

- $R(u) \equiv u$ if u is a constant;
- $R(a) \equiv u_1 \, a \, u_2$ if a is a positive atom;
- $R(b) \equiv u_2 \, a \, u_1$ if $b \equiv \bar{a}$ is a negative atom;
- $R(A \; \alpha \; B) \equiv R(A) \; \alpha \; R(B)$ for every relation α of $\mathscr{G}$.

For every formula $A \in \mathscr{F}$, $I(R(A)) \equiv A$.

Example 1.13. A natural interpretation for the set of subatomic formulae for classical logic defined in example 1.2 is given by considering the assignments:

$- \; I(\mathsf{t}) \equiv \mathsf{t}$;	$- \; I(\mathsf{f}) \equiv \mathsf{f}$;
$- \; \forall a \in \mathscr{A}. \; I(\mathsf{f} \; a \; \mathsf{f}) \equiv \mathsf{f}$;	$- \; \forall a \in \mathscr{A}. \; I(\mathsf{t} \; a \; \mathsf{t}) \equiv \mathsf{t}$;
$- \; \forall a \in \mathscr{A}. \; I(\mathsf{f} \; a \; \mathsf{t}) \equiv a$;	$- \; \forall a \in \mathscr{A}. \; I(\mathsf{t} \; a \; \mathsf{f}) \equiv \bar{a}$;
$- \; I(A \vee B) \equiv I(A) \vee I(B)$;	$- \; I(A \wedge B) \equiv I(A) \wedge I(B)$;

where $A, B \in \mathscr{F}^i$, and extending it in such a way that $A \; a \; B$ is interpretable iff $A = u, B = v$ with $u, v \in \{\mathsf{f}, \mathsf{t}\}$ and then $I(A \; a \; B) \equiv I(u \; a \; v)$.

For example, if $A \equiv (((\mathsf{f} \wedge \mathsf{t}) \; a \; \mathsf{t}) \vee \mathsf{t}) \wedge (\mathsf{t} \; b \; \mathsf{f})$, its interpretation is $I(A) = (a \vee \mathsf{t}) \wedge \bar{b}$.

Note that the set $\mathscr{F}^i$ of interpretable formulae is composed by all formulae equal to a formula where an atom does not occur in the scope of another atom. Every other formula is not interpretable, such as $B \equiv ((\mathsf{t} \; b \; \mathsf{f}) \wedge \mathsf{t}) \; a \; \mathsf{f}$.

Example 1.14. A natural interpretation for the set of subatomic formulae for multiplicative additive linear logic defined in example 1.3 is given by considering the assignments:

$- \; I(1) \equiv 1$;	$- \; I(\bot) \equiv \bot$;
$- \; \forall a \in \mathscr{A}. \; I(\bot \; a \; \bot) \equiv \bot$;	$- \; \forall a \in \mathscr{A}. \; I(1 \; a \; 1) \equiv 1$;
$- \; \forall a \in \mathscr{A}. \; I(\bot \; a \; 1) \equiv a$;	$- \; \forall a \in \mathscr{A}. \; I(1 \; a \; \bot) \equiv \bar{a}$;
$- \; I(A \parr B) \equiv I(A) \parr I(B)$;	$- \; I(A \otimes B) \equiv I(A) \otimes I(B)$;

where $A, B \in \mathscr{F}^i$, and extending it in such a way that $A \; a \; B$ is interpretable iff $A = u, B = v$ with $u, v \in \{\bot, 1\}$ and then $I(A \; a \; B) \equiv I(u \; a \; v)$.

For example, for $C \equiv ((1 \parr \bot) \; a \; 1) \otimes \bot$, $I(C) = a \otimes \bot$.

The formulae that are not interpretable are not only those equal to a formula where an atom occurs in the scope of another atom, but also those where a formula made up of units not equal to 1 or $\bot$ occurs in the scope of an atom, such as $(1 \parr 1) \; a \; \bot$.

Example 1.15. A natural interpretation for the set of subatomic formulae $\mathscr{F}_{bv}$ into the set of formulae of BVU is given by considering the assignments:

- $I(\bot) \equiv \bot$;
- $I(1) \equiv 1$;
- $I(\circ) \equiv \circ$;
- $\forall a \in \mathscr{A}.\ I(\bot\, a\, \bot) \equiv \bot$;
- $\forall a \in \mathscr{A}.\ I(1\, a\, 1) \equiv 1$;
- $\forall a \in \mathscr{A}.\ I(\bot\, a\, 1) \equiv a$;
- $\forall a \in \mathscr{A}.\ I(1\, a\, \bot) \equiv \bar{a}$;
- $I(A \parr B) \equiv I(A) \parr I(B)$;
- $I(A \otimes B) \equiv I(A) \otimes I(B)$;
- $I(A \lhd B) \equiv I(A) \lhd I(B)$;

where $A, B \in \mathscr{F}^i$, and extending it in such a way that $A\, a\, B$ is interpretable iff $A = u, B = v$ with $u, v \in \{\bot, 1\}$ and then $I(A\, a\, B) \equiv I(u\, a\, v)$.

The formulae that are not interpretable are not only those equal to a formula where an atom occurs in the scope of another atom, but also those where a formula made-up of units not equal to $\bot$ or 1 occurs in the scope of an atom, such as $(1 \parr 1)\, a\, \circ$.

This interpretation is also natural as an interpretation into the set of formulae of BV. Note that even though $\bot\, a\, 1 = \circ\, a\, 1$ in BV, the former is interpretable, while the latter is not. Interpretability is not necessarily preserved by equality.

1.2 Subatomic proof systems

The useful properties of subatomic formulae become apparent when we extend the principle to atomic inference rules. Let us consider, for example, the usual contraction rule for an atom in classical logic given by

$$\frac{a \vee a}{a} \quad .$$

We could obtain this rule subatomically through the interpretation map defined in example 1.13 as follows:

$$\frac{(\mathsf{f}\, a\, \mathsf{t}) \vee (\mathsf{f}\, a\, \mathsf{t})}{(\mathsf{f} \vee \mathsf{f})\, a\, (\mathsf{t} \vee \mathsf{t})} \overset{I}{\mapsto} \frac{a \vee a}{a} \quad \text{and} \quad \frac{(\mathsf{t}\, a\, \mathsf{f}) \vee (\mathsf{t}\, a\, \mathsf{f})}{(\mathsf{t} \vee \mathsf{t})\, a\, (\mathsf{f} \vee \mathsf{f})} \overset{I}{\mapsto} \frac{\bar{a} \vee \bar{a}}{\bar{a}} \quad .$$

These rules are therefore generated by the linear scheme

$$\frac{(A\, a\, B) \vee (C\, a\, D)}{(A \vee C)\, a\, (B \vee D)} \quad \text{, where } A, B, C, D \text{ are formulae.}$$

Strikingly, the non-linearity of the contraction rule has been pushed from the atoms to the units.

Similarly, we can consider the atomic identity rule

$$\frac{\mathsf{t}}{a \vee \bar{a}} \quad .$$

It can be obtained subatomically as follows:

$$\frac{(\mathsf{f} \vee \mathsf{t})\ a\ (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{f})} \overset{I}{\mapsto} \frac{\mathsf{t}}{a \vee \bar{a}} \quad .$$

Similarly to the contraction rule, it is generated by the linear scheme

$$\frac{(A \vee B)\ a\ (C \vee D)}{(A\ a\ C) \vee (B\ a\ D)} \quad \text{, where } A, B, C, D \text{ are formulae.}$$

It is quite plain to see that both the subatomic contraction rule and the subatomic introduction rule have the same shape. This surprising regularity is made very useful in combination with the observation that in fact the linear rule scheme

$$\frac{(A\ \alpha\ B)\ \nu\ (C\ \beta\ D)}{(A\ \nu\ C)\ \alpha\ (B\ \gamma\ D)} \quad ,$$

where $\alpha, \nu, \beta, \gamma$ are relations, and A, B, C, D are formulae is typical of logical rules in deep inference. We refer to it as a *medial shape*. For example, consider system SKS for classical logic:

$$\begin{array}{cc}
ai{\downarrow}\,\dfrac{\mathsf{t}}{a \vee \bar{a}} & ai{\uparrow}\,\dfrac{a \wedge \bar{a}}{\mathsf{f}} \\[2ex]
s\,\dfrac{(A \vee B) \wedge C}{(A \wedge C) \vee B} & m\,\dfrac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \\[2ex]
ac{\downarrow}\,\dfrac{a \vee a}{a} & ac{\uparrow}\,\dfrac{a}{a \wedge a} \\[2ex]
aw{\downarrow}\,\dfrac{\mathsf{f}}{a} & aw{\uparrow}\,\dfrac{a}{\mathsf{t}}
\end{array}$$

System SKS

We can see that the rule m follows this scheme as well, and we can derive the rule s from the rule

$$\wedge{\downarrow}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \quad ,$$

which follows this scheme. We have therefore uncovered an underlying sturucture behind the shape of inference rules, that we will exploit to obtain a general characterisation of rules.

To make use of the general characterisation, we will impose some restrictions on $\alpha, \nu, \beta, \gamma$. These conditions strike a balance between being general enough to encompass a wide variety of logics and being explicit enough to enable us to generalise procedure such as cut-elimination and decomposition. They are the result of a trial-and-error phase comprised of the comparison of different proof systems together with the study of the properties necessary for cut-elimination and decomposition results.

The restrictions on the relations of the rule scheme stem from the observation that certain dualities between the relations are maintained in every rule. For example, we can write the rule $\wedge\downarrow$ as

$$\wedge\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \mathbin{\bar{\wedge}} D)}$$

and the subatomic identity rule as

$$\frac{(A \vee B)\ a\ (C \vee D)}{(A\ a\ C) \vee (B\ \bar{a}\ D)} \quad .$$

We will generalise this observation, considering rules with a medial shape and certain dualities between the connectives involved and show that this shape is enough to represent a wide variety of logics. With the subatomic methodology, we are therefore able to represent proof systems in such a way that every rule has the same shape. This full regularity gives us a newly gained ability to characterise proof systems that enjoy properties such as decomposition and cut-elimination.

To characterise the dualities present in the inference rules, we introduce a notion of polarity in the pairs of dual relations. This notion of polarity can be reminiscent of the polarities assigned to connectives in linear logic [18], but the idea behind it is rather to assign which of the relations in the pair is 'stronger' than the other. Intuitively, it loosely corresponds to assigning which relation of the pair will imply the other. For example, in classical logic $A \wedge B$ implies $A \vee B$, and thus we will assign $\wedge$ to be *strong* and $\vee$ to be *weak*.

Definition 1.16. For each pair of relations $\{\alpha, \overline{\alpha}\}$, we give a polarity assignment: we call one relation of the pair *strong* and the other one *weak*.

If α is strong and $\overline{\alpha}$ is weak, we will write $\alpha^M{=}\overline{\alpha}^M{=}\alpha$ and $\alpha^m{=}\overline{\alpha}^m{=}\overline{\alpha}$. Self-dual relations are both strong and weak.

Definition 1.17. A *subatomic proof system* SA with set of formulae $\mathscr{F}$ is

- a collection of inference rules of the shape $\dfrac{(A\ \beta\ B)\ \ \alpha\ \ (C\ \beta\ \ D)}{(A\ \alpha\ C)\ \ \beta\ \ (B\ \alpha^m\ D)}$, $\alpha, \beta {\in}\ \mathscr{R}$, called *down-rules*,
- a collection of inference rules of the shape $\dfrac{(A\ \beta\ B)\ \ \ \alpha\ \ (C\ \beta^M\ D)}{(A\ \alpha\ C)\ \ \beta\ \ (B\ \alpha\ \ \ D)}$, $\alpha, \beta {\in}\ \mathscr{R}$, called *up-rules*,

- a collection of rules $= \frac{A}{B}$ and $= \frac{\overline{A}}{\overline{B}}$, for every axiom $A = B$ of the equational theory $=$ on $\mathscr{F}$, called *equality rules*.

Note that the non-invertible rules are linear: surprisingly, non-linearity can be pushed from the atoms to the units.

Remark 1.18. Since we will not always work modulo equality, we define the equality rules to be inference steps just like the inference rules, rather than focusing on equality as equations between formulae. Two formulae A and B will be equal if and only if there is a derivation from A to B composed only of equality rules.

We could have just as well defined equality between formulae directly in this way, but chose to define it initially as an equivalence relation for the sake of clearer exposition when defining the interpretation map.

The rules $= \frac{A}{B}$ are invertible and correspond to equivalence by mutual implication. Every non-invertible rule with logical significance is therefore an instance of the general rule scheme with medial shape.

Remark 1.19. We will often use the notation

$$\frac{(A \; \beta \; B) \; \alpha^M \; (C \; \beta \; D)}{(A \; \alpha \; B) \; \beta \; (C \; \overline{\alpha} \; D)}$$

for down-rules with a strong relation in the premiss where β is commutative.

Example 1.20. We consider $\wedge$ as strong and $\vee$ as weak in classical logic. The subatomic proof system SAKS is given by the inference rules in Figure 1-1, together with the equality rules given by $= \frac{A}{B}$ for every A, B on opposite sides of the equality axioms provided in example 1.7.

Rules labeled with a $\downarrow$ are down-rules, and rules labeled by a $\uparrow$ are up-rules. The medial rule labeled by m is self-dual, and is both a down-rule and an up-rule.

Example 1.21. We consider $\otimes$ as strong and $⅋$ as weak in multiplicative linear logic. The subatomic proof system SAMLLS is given by the inference rules in Figure 1-3 together with the equality rules given by $= \frac{A}{B}$ for every A, B on opposite sides of the equality axioms provided in example 1.8.

Example 1.22. We consider $\otimes$ as strong and $⅋$ as weak in BVU and BV. The subatomic proof system SABVU is given by the inference rules in Figure 1-5 together with the equality rules given by $= \frac{A}{B}$ for every A, B on opposite sides of the equality axioms for BVU provided in example 1.9.

$$a{\downarrow}\,\frac{(A \vee B)\; a\; (C \vee D)}{(A\; a\; C) \vee (B\; a\; D)} \qquad a{\uparrow}\,\frac{(A\; a\; B) \wedge (C\; a\; D)}{(A \wedge C)\; a\; (B \wedge D)}$$

$$\wedge{\downarrow}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \vee{\uparrow}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$m\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$ac\,\frac{(A\; a\; B) \vee (C\; a\; D)}{(A \vee C)\; a\; (B \vee D)} \qquad a\bar{c}\,\frac{(A \wedge B)\; a\; (C \wedge D)}{(A\; a\; C) \wedge (B\; a\; D)}$$

Figure 1-1: SAKS

$$i{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \qquad i{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}}$$

$$s\,\frac{(A \vee B) \wedge C}{(A \wedge C) \vee B} \qquad m\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$c{\downarrow}\,\frac{a \vee a}{a} \qquad c{\uparrow}\,\frac{a}{a \wedge a}$$

$$aw{\downarrow}\,\frac{\mathsf{f}}{a} \qquad aw{\uparrow}\,\frac{a}{\mathsf{t}}$$

Figure 1-2: SKS *[6]*

$$a{\downarrow}\,\frac{(A \parr B)\; a\; (C \parr D)}{(A\; a\; C) \parr (B\; a\; D)} \qquad a{\uparrow}\,\frac{(A\; a\; B) \otimes (C\; a\; D)}{(A \otimes C)\; a\; (B \otimes D)}$$

$$\otimes{\downarrow}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr{\uparrow}\,\frac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)}$$

Figure 1-3: SAMLLS

$$ai{\downarrow}\,\frac{1}{a \parr \bar{a}} \qquad ai{\uparrow}\,\frac{a \otimes \bar{a}}{\bot}$$

$$s\,\frac{(A \parr B) \otimes C}{(A \otimes C) \parr B}$$

Figure 1-4: SMLLS *[40]*

$$
\begin{array}{ll}
a{\downarrow}\,\dfrac{(A \parr B)\; a\; (C \parr D)}{(A\; a\; C) \parr (B\; a\; D)} & a{\uparrow}\,\dfrac{(A\; a\; B) \otimes (C\; a\; D)}{(A \otimes C)\; a\; (B \otimes D)} \\[2ex]
\otimes{\downarrow}\,\dfrac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} & \parr{\uparrow}\,\dfrac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)} \\[2ex]
\lhd{\downarrow}\,\dfrac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} & \lhd{\uparrow}\,\dfrac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)}
\end{array}
$$

Figure 1-5: SABV

$$
\begin{array}{ccc}
ai{\downarrow}\,\dfrac{\circ}{a \parr \bar{a}} & & a{\uparrow}\,\dfrac{a \otimes \bar{a}}{\circ} \\[2ex]
& s\,\dfrac{(A \parr B) \otimes C)}{(A \otimes C) \parr B} & \\[2ex]
q{\downarrow}\,\dfrac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} & & q{\uparrow}\,\dfrac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)}
\end{array}
$$

Figure 1-6: SBV *[26]*

Likewise, the subatomic proof system SABV is given by the same inference rules and equality rules, together with the equality rules given by $= \dfrac{\bot}{\circ}$, $= \dfrac{1}{\circ}$ and their converse.

Remark 1.23. An interesting future line of work is to characterise sound rules based on a partial order on relations. Some preliminary research in this direction has yielded very encouraging results. We assign a partial order based on implication to the relations of classical logic: $\vee < a < \wedge$.

Then, all down-rules in systems SAKS obey the scheme $\dfrac{(A\; \beta\; B)\quad \alpha\quad (C\; \beta\quad D)}{(A\; \alpha\; C)\quad \beta\quad (B\; \alpha^m\; D)}$, $\bar{\beta} \geq \alpha$.

Dually, all up-rules obey the scheme $\dfrac{(A\; \beta\; B)\quad \alpha\quad (C\; \beta^M\; D)}{(A\; \alpha\; C)\quad \beta\quad (B\; \alpha\quad D)}$, $\bar{\alpha} \geq \beta$.

Furthermore, every rule following this scheme is sound in classical logic.

We can similarly assign partial orders to the relations of multiplicative additive linear logic and BV ($\parr < \oplus < a < \& < \otimes$ and $\parr < \lhd, a < \otimes$). Then, the rules of systems SAMALLS (Figure 3-3) and SBV verify this scheme as well.

To reduce rules to their subatomic form, we will work in the setting of deep inference [23], where proofs can be composed with the same connectives as formulae. The deep inference methodology has been exploited in many ways, such as shortening analytic proofs by exponential factors with respect to Gentzen proofs [8, 13], modeling process algebras [7, 35, 37, 38] or typing optimised versions of the λ-calculus that provide a novel treatment of sharing and duplication [33]. The particular property that most interests us is that rules can be applied at any depth inside a formula and as a result every contraction and cut instances can be locally transformed into their atomic variants by a local procedure of polynomial-size complexity [6]. Therefore they can be transformed into their subatomic variants straightforwardly.

We will present proofs in the open deduction formalism [28], which is a logic-independent formalism, allowing us to reach the desired level of generality.

Definition 1.24. Given a subatomic systems SA and formulae A and B, a *derivation* ϕ in SA from *premiss* A to *conclusion* B denoted by $\boxed{\begin{array}{c} A \\ \phi \, \| \, \mathsf{SA} \\ B \end{array}}$ is defined to be:

- a formula $\phi \equiv A \equiv B$;
- a *composition by inference*

$$\phi \equiv \rho \, \frac{\boxed{\begin{array}{c} A \\ \phi_1 \, \| \, \mathsf{SA} \\ A' \end{array}}}{\boxed{\begin{array}{c} B' \\ \phi_2 \, \| \, \mathsf{SA} \\ B \end{array}}}$$

  where ρ is an instance of an inference rule in SA and ϕ_1 and ϕ_2 are derivations in SA;

- a *composition by relations*

$$\phi \equiv \boxed{\begin{array}{c} A_1 \\ \phi_1 \, \| \, \mathsf{SA} \\ B_1 \end{array}} \; \alpha \; \boxed{\begin{array}{c} A_2 \\ \phi_2 \, \| \, \mathsf{SA} \\ B_2 \end{array}}$$

  where $\alpha \in \mathscr{R}$, $A \equiv A_1 \; \alpha \; A_2$, $B \equiv B_1 \; \alpha \; B_2$, ϕ_1 and ϕ_2 are derivations in SA.

We denote by

$$\boxed{\begin{array}{c} A \\ \phi \, \| \, \{\rho_1,\dots,\rho_n\} \\ B \end{array}}$$

a derivation where only the rules $\rho_1, \dots, \rho_n$ appear.

Sometimes we omit the name of a derivation or the name of the proof system if there is no ambiguity.

To improve readability sometimes we remove the boxes around derivations.

Notation 1.25. We consider the two derivations

$$\rho_2\,\frac{\left(\begin{array}{c}A_1\\ \phi_1\,\Vert\,\mathsf{SA}\\ \rho_1\,\dfrac{A_2}{B_1}\\ \phi_2\,\Vert\,\mathsf{SA}\\ B_2\end{array}\right)}{\begin{array}{c}C_1\\ \phi_3\,\Vert\,\mathsf{SA}\\ C_2\end{array}} \qquad \text{and} \qquad \rho_1\,\frac{\begin{array}{c}A_1\\ \phi_1\,\Vert\,\mathsf{SA}\\ A_2\end{array}}{\left(\begin{array}{c}B_1\\ \phi_2\,\Vert\,\mathsf{SA}\\ \rho_2\,\dfrac{B_2}{C_1}\\ \phi_3\,\Vert\,\mathsf{SA}\\ C_2\end{array}\right)}$$

to be equal and we denote them both by

$$\begin{array}{c}A_1\\ \phi_1\,\Vert\,\mathsf{SA}\\ \rho_1\,\dfrac{A_2}{B_1}\\ \phi_2\,\Vert\,\mathsf{SA}\\ \rho_2\,\dfrac{B_2}{C_1}\\ \phi_2\,\Vert\,\mathsf{SA}\\ C_2\end{array} .$$

Example 1.26. The following is a SAKS derivation with premiss $(\mathsf{f}\vee\mathsf{t})a(\mathsf{t}\vee\mathsf{f})\wedge((\mathsf{f}b\mathsf{t})\vee\mathsf{t})\wedge\mathsf{t}$ and conclusion $((\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{f}\;b\;\mathsf{t}))\vee((\mathsf{t}\;a\;\mathsf{f})\vee\mathsf{t})\wedge\mathsf{t}$:

$$\boxed{s\,\dfrac{\boxed{ai{\downarrow}\,\dfrac{(\mathsf{f}\vee\mathsf{t})\;a\;(\mathsf{t}\vee\mathsf{f})}{(\mathsf{f}\;a\;\mathsf{t})\vee(\mathsf{t}\;a\;\mathsf{f})}}\wedge((\mathsf{f}\;b\;\mathsf{t})\vee\mathsf{t})}{((\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{f}\;b\;\mathsf{t}))\vee((\mathsf{t}\;a\;\mathsf{f})\vee\mathsf{t})}}\wedge\mathsf{t}$$

Definition 1.27. Let $\begin{array}{c}A\\ \phi\,\Vert\,\mathsf{SA}\\ B\end{array}$ and $\begin{array}{c}B\\ \psi\,\Vert\,\mathsf{SA}\\ C\end{array}$ be two derivations. We define their *composition* $\begin{array}{c}\phi\\ \cdots\\ \psi\end{array}$ as the derivation constructed as follows:

- if ϕ is a formula then $\begin{array}{c}\phi\\ \cdots\\ \psi\end{array}\equiv\psi$; likewise if ψ is a formula then $\begin{array}{c}\phi\\ \cdots\\ \psi\end{array}\equiv\phi$;

- if $\phi\equiv\dfrac{\phi_1}{\phi_2}$ then $\begin{array}{c}\phi\\ \cdots\\ \psi\end{array}\equiv\dfrac{\phi_1}{\left(\begin{array}{c}\phi_2\\ \cdots\\ \psi\end{array}\right)}$; likewise if $\psi\equiv\dfrac{\psi_1}{\psi_2}$ then $\begin{array}{c}\phi\\ \cdots\\ \psi\end{array}\equiv\dfrac{\left(\begin{array}{c}\phi\\ \cdots\\ \psi_1\end{array}\right)}{\psi_2}$;

- if $\phi \equiv \phi_1 \; \alpha \; \phi_2$ and $\psi \equiv \psi_1 \; \alpha \; \psi_2$ then $\begin{array}{c}\phi\\ \cdots\\ \psi\end{array} \equiv \begin{array}{c}\phi_1\\ \cdots\\ \psi_1\end{array} \; \alpha \; \begin{array}{c}\phi_2\\ \cdots\\ \psi_2\end{array}$.

Definition 1.28. Let $\begin{array}{c}A\\ \phi \, \| \, \mathsf{SA}\\ B\end{array}$ be a derivation, and $K\{\ \}$ a context. We define the derivation $K\{\phi\}$ from $K\{A\}$ to $K\{B\}$ as the derivation obtained by inserting ϕ in the place of the hole in $K\{\ \}$.

Example 1.29. If $\phi = ai\!\downarrow \dfrac{(\mathsf{f} \vee \mathsf{t}) \; a \; (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f} \; a \; \mathsf{t}) \vee (\mathsf{t} \; a \; \mathsf{f})}$ and $K\{\ \} = (\mathsf{t} \wedge \{\ \}) \vee (\mathsf{f} \wedge \mathsf{f})$, then

$$K\{\phi\} = \left(\mathsf{t} \wedge ai\!\downarrow \frac{(\mathsf{f} \vee \mathsf{t}) \; a \; (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f} \; a \; \mathsf{t}) \vee (\mathsf{t} \; a \; \mathsf{f})}\right) \vee (\mathsf{f} \wedge \mathsf{f}) \quad .$$

Sometimes we will work by induction on the number of rules on a derivation. For that, it is useful to impose an order on the rules to have a notion of which one is the 'last' rule of the derivation. We impose this order by sequentialising the derivation.

Definition 1.30. Let $\begin{array}{c}A\\ \phi \, \|\\ B\end{array}$ be a derivation. We define the sequential form of ϕ as follows by structural induction on ϕ:

- if $\phi \equiv A$ is a formula, then its sequential form is given by A ;

- if $\phi \equiv \rho \dfrac{\boxed{\begin{array}{c}A\\ \phi_1 \, \|\\ A'\end{array}}}{\boxed{\begin{array}{c}B'\\ \phi_2 \, \|\\ B\end{array}}}$, then we consider ϕ_1 and ϕ_2 in sequential form:

$$\phi_1 = \begin{array}{c} \rho_1 \dfrac{A}{A_2} \\ \vdots \\ \rho_n \dfrac{A_n}{A'} \end{array} \qquad \text{and} \qquad \phi_2 = \begin{array}{c} \rho_{n+1} \dfrac{B'}{B_2} \\ \vdots \\ \rho_m \dfrac{B_m}{B} \end{array}$$

and the sequential form of ϕ is given by

$$\phi = \begin{array}{r}A\\ \rho_1 \overline{\overline{}}\\ \vdots\\ \rho_n \overline{\overline{}}\\ A'\\ \rho \overline{\overline{}}\\ B'\\ \rho_{n+1} \overline{\overline{}}\\ \vdots\\ \rho_{n+m} \overline{\overline{}}\\ B\end{array} .$$

- if $\phi \equiv \boxed{\begin{array}{c}A_1\\ \phi_1 \Vert\\ B_1\end{array}} \; \alpha \; \boxed{\begin{array}{c}A_2\\ \phi_2 \Vert\\ B_2\end{array}}$, then we sequentialise ϕ_1 and ϕ_2 to obtain

$$\phi_1 = \begin{array}{c} \rho_1 \dfrac{A_1}{C_2} \\ \vdots \\ \rho_n \dfrac{C_n}{B_1} \end{array} \qquad \text{and} \qquad \phi_2 = \begin{array}{c} \rho_{n+1} \dfrac{A_2}{D_2} \\ \vdots \\ \rho_{n+m} \dfrac{D_m}{B_2} \end{array}$$

and the sequential form of ϕ is given by

$$\phi = \begin{array}{c} \rho_1 \dfrac{A_1 \; \alpha \; A_2}{C_2 \; \alpha \; A_2} \\ \vdots \\ \rho_n \dfrac{C_n \; \alpha \; A_2}{B_1 \; \alpha \; A_2} \\ \rho_{n+1} \dfrac{}{B_1 \; \alpha \; D_2} \\ \vdots \\ \rho_{n+m} \dfrac{B_1 \; \alpha \; D_m}{B_1 \; \alpha \; B_2} \end{array} .$$

To simplify readability, when there is no ambiguity we will represent the sequential form through single lines $\dfrac{A}{B}$ instead of double lines $\overset{A}{\underset{B}{=}}$.

The sequential form is not a normal form: we can choose how to sequentialise a composition by relation, by starting from either side of the relation. However we make this choice, the number of rules in the sequential form of the derivation stays nonetheless equal to the number of inference rules in its open deduction form.

Example 1.31. The sequential form of the derivation ϕ of example 1.26 is:

$$\phi = \begin{array}{l} \text{ai}\!\downarrow\dfrac{(((\mathsf{f} \vee \mathsf{t})\ a\ (\mathsf{t} \vee \mathsf{f})) \wedge ((\mathsf{f}\ b\ \mathsf{t}) \vee \mathsf{t})) \wedge \mathsf{t}}{s\,\dfrac{(((\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{f})) \wedge ((\mathsf{f}\ b\ \mathsf{t}) \vee \mathsf{t})) \wedge \mathsf{t}}{(((\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{f}\ b\ \mathsf{t})) \vee ((\mathsf{t}\ a\ \mathsf{f}) \vee \mathsf{t})) \wedge \mathsf{t}}} \end{array} .$$

For some results, such as the splitting theorem in Section 2 it is useful to consider proofs modulo certain equalities. To simplify the presentation and the case analysis, we define the Calculus of Structures presentation. This presentation provides us with a natural way of extending an equivalence relation between formulae to an equivalence relation between derivations.

Definition 1.32. Let $\sim$ be an equivalence relation on $\mathscr{F}$ obtained from a subset of the axioms that define $=$ as per Definition 1.5.

If $C \sim C'$, there is a derivation $\begin{array}{c} C \\ \zeta \| \\ C' \end{array}$ where ζ is composed only of equality rules corresponding to the axioms of $\sim$. We will denote such derivations by $\sim \frac{C}{\overline{C'}}$.

A derivation in sequential form

$$\phi = \begin{array}{c} \sim \dfrac{A_0}{\overline{A_1}} \\ \rho_1 \dfrac{A_1}{A_2} \\ \sim \dfrac{A_2}{\overline{A_3}} \\ \vdots \\ \sim \dfrac{A_n}{\overline{A_{n+1}}} \\ \rho_m \dfrac{A_{n+1}}{A_{n+2}} \\ \sim \dfrac{A_{n+2}}{\overline{A_{n+3}}} \end{array}$$

has *Calculus of Structures (CoS) notation* for $\sim$ given by

$$\phi = \begin{array}{c} \rho_1 \dfrac{A_0}{A_3} \\ \vdots \\ \rho_m \dfrac{A_{n+1}}{A_{n+3}} \end{array} .$$

We define the equivalence relation $\sim$ on derivations as $\phi_1 \sim \phi_2$ if

$$\phi_1 = \begin{array}{c} \rho_1 \dfrac{A_0}{A_1} \\ \vdots \\ \rho_{n+1} \dfrac{A_n}{A_{n+1}} \end{array} \quad \text{and} \quad \phi_2 = \begin{array}{c} \rho_1 \dfrac{A'_0}{A'_1} \\ \vdots \\ \rho_{n+1} \dfrac{A'_n}{A'_{n+1}} \end{array}$$

in CoS notation for $\sim$, with $A_i \sim A'_i$ for every $0 \leq i \leq n+1$.

Example 1.33. If $\sim$ is the equivalence relation on the set of formulae $\mathscr{F}_{cl}$ for classical logic defined by the axiom $A \wedge \mathsf{t} = A$, then

$$\begin{array}{c} ai\!\downarrow \dfrac{(((\mathsf{f} \vee \mathsf{t})\; a\; (\mathsf{t} \vee \mathsf{f})) \wedge ((\mathsf{f}\; b\; \mathsf{t}) \vee \mathsf{t})) \wedge \mathsf{t}}{s\, \dfrac{(((\mathsf{f}\; a\; \mathsf{t}) \vee (\mathsf{t}\; a\; \mathsf{f})) \wedge ((\mathsf{f}\; b\; \mathsf{t}) \vee \mathsf{t})) \wedge \mathsf{t}}{(((\mathsf{f}\; a\; \mathsf{t}) \wedge (\mathsf{f}\; b\; \mathsf{t})) \vee ((\mathsf{t}\; a\; \mathsf{f}) \vee \mathsf{t})) \wedge \mathsf{t}}} \end{array} \sim \begin{array}{c} ai\!\downarrow \dfrac{((\mathsf{f} \vee \mathsf{t})\; a\; (\mathsf{t} \vee \mathsf{f})) \wedge ((\mathsf{f}\; b\; \mathsf{t}) \vee \mathsf{t})}{s\, \dfrac{((\mathsf{f}\; a\; \mathsf{t}) \vee (\mathsf{t}\; a\; \mathsf{f})) \wedge ((\mathsf{f}\; b\; \mathsf{t}) \vee \mathsf{t})}{((\mathsf{f}\; a\; \mathsf{t}) \wedge (\mathsf{f}\; b\; \mathsf{t})) \vee ((\mathsf{t}\; a\; \mathsf{f}) \vee \mathsf{t})}} \end{array} .$$

1.3 Proofs

To study proof theory through subatomic proof systems, we need to have a notion of proofs equivalent to that of the 'regular' theory. For that, we will establish a notion of correspondence between subatomic systems and deep inference systems. In a *correct* proof system every 'ordinary' proof will have a corresponding subatomic proof, and every subatomic proof where every step has an interpretation will correspond to an 'ordinary' proof.

Definition 1.34. Let $1 \in \mathscr{U}$ be a distinguished constant. A *proof* of A is a derivation ϕ whose premiss is 1. We denote proofs by $\begin{array}{c} \phi \| \\ A \end{array}$.

For reasons of convention, the distinguished unit for each proof system might be denoted with a different symbol, as is the case for classical logic.

Example 1.35. A proof in SAKS is a derivation with premiss t.

Example 1.36. A proof in SAMLLS is a derivation with premiss 1.

Example 1.37. A proof in SABV is a derivation with premiss 1.

Definition 1.38. Given an interpretation map I for SA, a derivation is *interpretable* if every formula appearing in its sequential form is interpretable.

Definition 1.39. Let SA be a subatomic system with a natural interpretation I into the set $\mathscr{G}$ of formulae of a complete proof system S for a propositional logic L, with associated representation map R.

We say that SA is *correct* for S when:

- for every interpretable SA derivation ψ with premiss P and conclusion C, there is a derivation ψ' in S with premiss $I(P)$ and conclusion $I(C)$; and

- for every derivation ϕ in S with premiss P' and conclusion C', there is an interpretable derivation ϕ' in SA with premiss $R(P')$ and conclusion $R(C')$.

Lemma 1.40. *Let* SA *be a subatomic system with a natural interpretation* I *into the set* $\mathscr{G}$ *of formulae of a complete proof system* S *for a propositional logic* L*, with associated representation map* R*.*

SA *is correct for* S *if, and only if:*

- *for every interpretable instance of an inference rule of* SA

$$\rho \frac{A}{B} \qquad ,$$

there is a derivation

$$\begin{array}{c} I(A) \\ \| \mathsf{S} \\ I(B) \end{array} \qquad ;$$

- *for every interpretable instance of derivations of the form*

$$\rho \frac{A}{B}\, a\, C \qquad \textit{and} \qquad D\, a\, \rho \frac{A}{B}$$

with $a \in \mathscr{A}$ *and* ρ *an inference rule of* SA*, there are derivations*

$$\begin{array}{c} I(A\, a\, C) \\ \| \mathsf{S} \\ I(B\, a\, C) \end{array} \qquad \textit{and} \qquad \begin{array}{c} I(D\, a\, A) \\ \| \mathsf{S} \\ I(D\, a\, B) \end{array} \qquad ; \textit{ and}$$

- *for every inference rule*

$$r \frac{A}{B}$$

of S*, there is an interpretable derivation*

$$\begin{array}{c} R(A) \\ \| \mathsf{SA} \\ R(B) \end{array} \qquad .$$

Proof. It is clear from how derivations are built and from the fact that $I(A\ \alpha\ B) = I(A)\ \alpha\ I(B)$ for $\alpha \in \mathscr{R}'$ and that $R(A\ \alpha\ B) = R(A)\ \alpha\ R(B)$ for $\alpha \in \mathscr{R}'$.

□

Example 1.41. System SAKS of Figure 1-1 is correct for system SKS of Figure 1-2.

Every interpretable assignment of units in the inference rules has a corresponding derivation in SKS. For example, for rule $a\!\downarrow$ we have the following interpretable assignments:

$$
\begin{array}{ll}
a{\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{t})\,a\,(\mathsf{t}\vee\mathsf{t})}{(\mathsf{t}\,a\,\mathsf{t})\vee(\mathsf{t}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}\vee\mathsf{t}} & a{\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{f})\,a\,(\mathsf{f}\vee\mathsf{f})}{(\mathsf{f}\,a\,\mathsf{f})\vee(\mathsf{f}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{\mathsf{f}\vee\mathsf{f}} \\
a{\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{t})\,a\,(\mathsf{t}\vee\mathsf{f})}{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{a\vee\bar{a}} & a{\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{f})\,a\,(\mathsf{f}\vee\mathsf{t})}{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{f}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\bar{a}\vee a} \\
a{\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{t})\,a\,(\mathsf{f}\vee\mathsf{t})}{(\mathsf{f}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{f}\vee\mathsf{t}} & a{\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{f})\,a\,(\mathsf{t}\vee\mathsf{f})}{(\mathsf{t}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}\vee\mathsf{f}} \\
a{\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{f})\,a\,(\mathsf{t}\vee\mathsf{t})}{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{a}{a\vee a} & a{\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{t})\,a\,(\mathsf{f}\vee\mathsf{f})}{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}\vee\bar{a}} \\
a{\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{t})\,a\,(\mathsf{t}\vee\mathsf{t})}{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{t}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{a\vee\mathsf{t}} & a{\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{t})\,a\,(\mathsf{f}\vee\mathsf{t})}{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\bar{a}\vee\mathsf{t}} \\
a{\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{f})\,a\,(\mathsf{t}\vee\mathsf{t})}{(\mathsf{t}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}\vee a} & a{\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{t})\,a\,(\mathsf{t}\vee\mathsf{f})}{(\mathsf{t}\,a\,\mathsf{t})\vee(\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}\vee\bar{a}} \\
a{\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{t})\,a\,(\mathsf{f}\vee\mathsf{f})}{(\mathsf{f}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\bar{a}}{\mathsf{f}\vee\bar{a}} & a{\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{f})\,a\,(\mathsf{f}\vee\mathsf{t})}{(\mathsf{f}\,a\,\mathsf{f})\vee(\mathsf{f}\,a\,\mathsf{t})} \overset{I}{\mapsto} \dfrac{a}{\mathsf{f}\vee a} \\
a{\downarrow}\dfrac{(\mathsf{t}\vee\mathsf{f})\,a\,(\mathsf{f}\vee\mathsf{f})}{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{f}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}\vee\mathsf{f}} & a{\downarrow}\dfrac{(\mathsf{f}\vee\mathsf{f})\,a\,(\mathsf{t}\vee\mathsf{f})}{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{f})} \overset{I}{\mapsto} \dfrac{a}{a\vee\mathsf{f}} \quad .
\end{array}
$$

It is easy to see that for each of them there is an SKS derivation with the same premiss and conclusion as the interpretation.

Likewise, we can check every interpretable instance of a rule inside the scope of an atom:

$$
\begin{array}{ll}
\dfrac{\mathsf{f}}{\mathsf{f}}\,a\,\mathsf{f} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{\mathsf{f}} & \dfrac{\mathsf{f}}{\mathsf{f}}\,a\,\mathsf{t} \overset{I}{\mapsto} \dfrac{a}{a} \\
\dfrac{\mathsf{t}}{\mathsf{t}}\,a\,\mathsf{f} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}} & \dfrac{\mathsf{t}}{\mathsf{t}}\,a\,\mathsf{t} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}} \\
\mathsf{f}\,a\,\dfrac{\mathsf{f}}{\mathsf{f}} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{\mathsf{f}} & \mathsf{t}\,a\,\dfrac{\mathsf{f}}{\mathsf{f}} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}} \\
\mathsf{f}\,a\,\dfrac{\mathsf{t}}{\mathsf{t}} \overset{I}{\mapsto} \dfrac{a}{a} & \mathsf{t}\,a\,\dfrac{\mathsf{t}}{\mathsf{t}} \overset{I}{\mapsto} \dfrac{\mathsf{t}}{\mathsf{t}} \\
\dfrac{\mathsf{f}}{\mathsf{t}}\,a\,\mathsf{f} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{\bar{a}} & \dfrac{\mathsf{f}}{\mathsf{t}}\,a\,\mathsf{t} \overset{I}{\mapsto} \dfrac{a}{\mathsf{t}} \\
\mathsf{f}\,a\,\dfrac{\mathsf{f}}{\mathsf{t}} \overset{I}{\mapsto} \dfrac{\mathsf{f}}{a} & \mathsf{t}\,a\,\dfrac{\mathsf{f}}{\mathsf{t}} \overset{I}{\mapsto} \dfrac{\bar{a}}{\mathsf{t}} \quad .
\end{array}
$$

It is easy to see that for each of them there is an SKS derivation with the same premiss and conclusion as the interpretation.

Furthermore, every inference rule of system SAKS trivially corresponds to the representation of an inference rule of system SKS , except for the rules $aw\downarrow$ and $aw\uparrow$.

$aw\downarrow$ corresponds to

$$
=\frac{\mathsf{f}}{=\dfrac{\mathsf{f}\ a\ \boxed{m\,\dfrac{(\mathsf{f}\wedge\mathsf{t})\vee(\mathsf{t}\wedge\mathsf{f})}{(\mathsf{f}\vee\mathsf{t})\wedge(\mathsf{t}\vee\mathsf{f})}}}{\mathsf{f}\ a\ \mathsf{t}}} \overset{I}{\mapsto} \frac{\mathsf{f}}{a} \quad \text{and} \quad =\frac{\mathsf{f}}{=\dfrac{\boxed{m\,\dfrac{(\mathsf{f}\wedge\mathsf{t})\vee(\mathsf{t}\wedge\mathsf{f})}{(\mathsf{f}\vee\mathsf{t})\wedge(\mathsf{t}\vee\mathsf{f})}}\ a\ \mathsf{f}}{\mathsf{t}\ a\ \mathsf{f}}} \overset{I}{\mapsto} \frac{\mathsf{f}}{\bar{a}} \quad ,
$$

and $aw\uparrow$ is the image of the dual derivations.

Furthermore, $\vee$ and $\wedge$ are associative and commutative in SAKS and their units are f and t respectively, and so the conditions are trivially verified for the equality inference rules.

Example 1.42. System SAMLLS of Figure 1-3 is correct for the multiplicative fragment of system SLLS given in Figure 1-4..

Every interpretable assignment of units in the inference rules has a corresponding derivation in the multiplicative fragment of SLLS. For example, for rule $a\downarrow$ we have the following interpretable assignments:

$$
\begin{array}{ll}
a\!\downarrow\dfrac{(\bot\parr\bot)\ a\ (\bot\parr\bot)}{(\bot\ a\ \bot)\parr(\bot\ a\ \bot)} \overset{I}{\mapsto} \dfrac{\bot}{\bot\parr\bot} & \\
a\!\downarrow\dfrac{(\bot\parr 1)\ a\ (1\parr\bot)}{(\bot\ a\ 1)\parr(1\ a\ \bot)} \overset{I}{\mapsto} \dfrac{1}{a\parr\bar{a}} & a\!\downarrow\dfrac{(1\parr\bot)\ a\ (\bot\parr 1)}{(1\ a\ \bot)\parr(\bot\ a\ 1)} \overset{I}{\mapsto} \dfrac{1}{\bar{a}\parr a} \\
a\!\downarrow\dfrac{(\bot\parr 1)\ a\ (\bot\parr 1)}{(\bot\ a\ \bot)\parr(1\ a\ 1)} \overset{I}{\mapsto} \dfrac{1}{\bot\parr 1} & a\!\downarrow\dfrac{(1\parr\bot)\ a\ (1\parr\bot)}{(1\ a\ 1)\parr(\bot\ a\ \bot)} \overset{I}{\mapsto} \dfrac{1}{1\parr\bot} \\
a\!\downarrow\dfrac{(\bot\parr\bot)\ a\ (\bot\parr 1)}{(\bot\ a\ \bot)\parr(\bot\ a\ 1)} \overset{I}{\mapsto} \dfrac{a}{\bot\parr a} & a\!\downarrow\dfrac{(\bot\parr 1)\ a\ (\bot\parr\bot)}{(\bot\ a\ \bot)\parr(1\ a\ \bot)} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bot\parr\bar{a}} \\
a\!\downarrow\dfrac{(\bot\parr\bot)\ a\ (1\parr\bot)}{(\bot\ a\ 1)\parr(\bot\ a\ \bot)} \overset{I}{\mapsto} \dfrac{a}{a\parr\bot} & a\!\downarrow\dfrac{(1\parr\bot)\ a\ (\bot\parr\bot)}{(1\ a\ \bot)\parr(\bot\ a\ \bot)} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a}\parr\bot} \quad .
\end{array}
$$

It is easy to see that for each of them there is a derivation in the multiplicative fragment of SLLS with the same premiss and conclusion as the interpretation.

Every interpretable instance of a rule ρ inside the scope of an atom is necessarily an instance where the premiss and conclusion of ρ are interpreted as constants. The only such instances are of the form $\dfrac{u}{u}$ with $u \in \{\bot, 1\}$ and therefore every interpretable instance of a rule inside the scope of an atom trivially corresponds to a derivation in the multiplicative fragment of SLLS.

Every inference rule of SAMLLS of Figure 1-3 trivially corresponds to the representation of an inference rule of the multiplicative fragment of system SLLS.

$\parr$ and $\otimes$ are associative and commutative in SAMLLS and their units are $\bot$ and 1

respectively. Therefore, the equality rules trivially verify the conditions.

Example 1.43. System SABV of Figure 1-5 is correct for system SBV given in Figure 1-6.

Every interpretable assignment of units in the inference rules has a corresponding derivation in SBV. For example, for rule $a\downarrow$ we have the following interpretable assignments:

$$
\begin{array}{ll}
a\downarrow \dfrac{(\bot \mathbin{⅋} \bot)\; a\; (\bot \mathbin{⅋} \bot)}{(\bot\; a\; \bot) \mathbin{⅋} (\bot\; a\; \bot)} \overset{I}{\mapsto} \dfrac{\bot}{\bot \mathbin{⅋} \bot} & \\
a\downarrow \dfrac{(\bot \mathbin{⅋} 1)\; a\; (1 \mathbin{⅋} \bot)}{(\bot\; a\; 1) \mathbin{⅋} (1\; a\; \bot)} \overset{I}{\mapsto} \dfrac{1}{a \mathbin{⅋} \bar{a}} & a\downarrow \dfrac{(1 \mathbin{⅋} \bot)\; a\; (\bot \mathbin{⅋} 1)}{(1\; a\; \bot) \mathbin{⅋} (\bot\; a\; 1)} \overset{I}{\mapsto} \dfrac{1}{\bar{a} \mathbin{⅋} a} \\
a\downarrow \dfrac{(\bot \mathbin{⅋} 1)\; a\; (\bot \mathbin{⅋} 1)}{(\bot\; a\; \bot) \mathbin{⅋} (1\; a\; 1)} \overset{I}{\mapsto} \dfrac{1}{\bot \mathbin{⅋} 1} & a\downarrow \dfrac{(1 \mathbin{⅋} \bot)\; a\; (1 \mathbin{⅋} \bot)}{(1\; a\; 1) \mathbin{⅋} (\bot\; a\; \bot)} \overset{I}{\mapsto} \dfrac{1}{1 \mathbin{⅋} \bot} \\
a\downarrow \dfrac{(\bot \mathbin{⅋} \bot)\; a\; (\bot \mathbin{⅋} 1)}{(\bot\; a\; \bot) \mathbin{⅋} (\bot\; a\; 1)} \overset{I}{\mapsto} \dfrac{a}{\bot \mathbin{⅋} a} & a\downarrow \dfrac{(\bot \mathbin{⅋} 1)\; a\; (\bot \mathbin{⅋} \bot)}{(\bot\; a\; \bot) \mathbin{⅋} (1\; a\; \bot)} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bot \mathbin{⅋} \bar{a}} \\
a\downarrow \dfrac{(\bot \mathbin{⅋} \bot)\; a\; (1 \mathbin{⅋} \bot)}{(\bot\; a\; 1) \mathbin{⅋} (\bot\; a\; \bot)} \overset{I}{\mapsto} \dfrac{a}{a \mathbin{⅋} \bot} & a\downarrow \dfrac{(1 \mathbin{⅋} \bot)\; a\; (\bot \mathbin{⅋} \bot)}{(1\; a\; \bot) \mathbin{⅋} (\bot\; a\; \bot)} \overset{I}{\mapsto} \dfrac{\bar{a}}{\bar{a} \mathbin{⅋} \bot} \quad .
\end{array}
$$

It is easy to see that for each of them there is a derivation in SBV with the same premiss and conclusion as the interpretation.

Every interpretable inference rule in the scope of an atom corresponds to a rule $\dfrac{u}{u}$ with $u \in \{\bot, \circ, 1\}$ and therefore trivially corresponds to an SBV derivation.

Every inference rule of system SABV is trivially the representation of an inference rule of system SBV, and the equality axioms are trivially represented by the equational theory for SABV we defined in example 1.9 where the units are identified.

In the next chapter we will focus on showing the admissibility of certain distinguished rules.

Definition 1.44. We say that an inference rule ρ is *admissible* for a proof system SA if $\rho \notin \mathsf{SA}$ and for every proof $\overset{\Pi\, \mathsf{SA} \cup \{\rho\}}{A}$ there exists a proof $\overset{\Pi\, \mathsf{SA}}{A}$.

Chapter 2

Splitting

Cut-elimination via splitting has been shown to work in a vast array of deep inference systems: linear logic [39], multiplicative exponential linear logic [41], the mixed commutative/non-commutative logic BV [26] and its extension with linear exponentials NEL [31] and classical predicate logic [4]. This generality points towards the fact that the splitting procedure hinges on some fundamental properties required for cut-elimination rather than on the specificities of each system.

In particular, cut-elimination proofs via splitting are very straightforward in those systems without contractions, as we will show in Section 2.1 with the example of multiplicative linear logic. This suggests that it is the properties of linear rules (as opposed to contraction rules) that enable us to prove cut-elimination. Indeed, the generalisation of the splitting procedure that we show in Section 2.2 allows us to fully confirm these suspicions: it is precisely because of the properties of the linear rules that we are able to prove cut-elimination for systems where they are present. In this way, we will give sufficient conditions that guarantee cut-elimination for a full class of substructural logics, similarly to [2, 43, 20] where conditions for a display calculus to enjoy cut elimination are presented, or to [36] where conditions for propositional based logics in the sequent calculus are presented.

2.1 Splitting for MLL

Linear logic was developed by Girard [19] as a refinement of classical logic by introducing restrictions on the structural rules of contraction and weakening. The core propositional connectives of linear logic are divided into additive and multiplicative connectives, exemplifying perfectly the distinction we will be making in this thesis between contractive systems and linear systems (that we will call *splittable*). The introduction rules for the additive conjunction $\&$ (with) and the multiplicative conjunction $\otimes$ (tensor) are given in their sequent calculus presentation as follows:

$$\frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi} \quad , \qquad \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \quad .$$

$$a{\downarrow}\,\frac{(A \parr B)\; a\; (C \parr D)}{(A\; a\; C) \parr (B\; a\; D)}$$

$$\otimes{\downarrow}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)}$$

Figure 2-1: *System* SAMLLS$^{\downarrow}$

Reading bottom up, we see that the additive conjunction & requires a duplication of the context whereas the multiplicative conjunction $\otimes$ requires that the context be divided between its hypotheses. There is no communication between Φ and Ψ in the proof above the tensor rule where they are united.

$$\begin{array}{c}
\begin{array}{cc} \Pi_1 & \Pi_2 \end{array} \\
\otimes\,\dfrac{\vdash A, \Phi \qquad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \\
\Delta \\
\vdash F\{A \otimes B\}, \Gamma
\end{array}$$

It is precisely this multiplicative rule shape that splitting hinges on. In the sequent calculus, the presence of a main connective allows us to know exactly which rules can be applied above a cut. In deep inference, this is not possible since any rule can be applied at any depth, and we therefore focus on the behaviour of the context around a cut to tackle cut-elimination. This allows us to have a better understanding of how the cut-elimination procedure changes the proof globally. If all the connectives of a system require a splitting of the context like the multiplicative tensor does, then we can keep track of exactly how the context around a connective behaves. This allows us to split a proof into independent subproofs above every rule, just like in the example above the proof is divided into Π_1 and Π_2 above the $\otimes$ introduction rule. Cut-elimination is then only a matter of rearranging the independent subproofs into a cut-free proof.

Multiplicative linear logic (MLL) is the fragment of linear logic comprising only the multiplicative connectives and their units. It is a very simple system in which every connective requires such a splitting of the context, and therefore ideal to provide an example of a proof of cut-elimination via splitting. In what follows we will present a proof of cut-elimination via splitting for MLL, as an example of an application of the generalised theorem of Section 2.2.

We will present this proof in the subatomic proof system for multiplicative linear logic SAMLLS to help the reader become accustomed to the subatomic notation, as well as to relate it better to the generalised theorem. We present subatomic system SAMLLS for MLL in Figure 1-3, together with the equations of example 1.8 and the interpretation map in example 1.14.

As is usual in deep inference systems, the sequent calculus cut rule is divided into several rules, corresponding to the up rules indicated by $\uparrow$. The splitting method allows us to prove the admissibility of all of these rules. The reduced cut-free system is denoted by $\mathsf{SAMLLS}^{\downarrow}$, and is shown in Figure 2-1.

By simple observation, we can notice that in $\mathsf{SAMLLS}^{\downarrow}$ the scope of the relations a and $\otimes$ only decreases when reading top to bottom. The widening scope of relations from bottom to top is the main property used to prove splitting. If we follow a particular instance of the tensor $\otimes$ through a proof, its scope will be at its widest in the premiss. Therefore, if we have a proof of $F\{A \otimes B\}$, we can follow $\otimes$ up in the proof to obtain two independent proofs $\begin{array}{c} \Pi_1 \Vert \\ Q_A\{A\} \end{array}$ and $\begin{array}{c} \Pi_2 \Vert \\ Q_B\{B\} \end{array}$.

$$\frac{\boxed{\begin{array}{c} \Pi_1 \Vert \\ A \parr K_1 \parr Q_1 \end{array}} \otimes \boxed{\begin{array}{c} \Pi_2 \Vert \\ B \parr K_2 \parr Q_2 \end{array}}}{\boxed{\begin{array}{c} (A \parr K_1) \otimes (B \parr K_2) \\ \Vert \\ (A \otimes B) \parr K_1 \parr K_2 \end{array}} \parr Q_1 \parr Q_2}$$

If we do this for every occurrence of $\otimes$ and a in the conclusion of a proof, starting from the outermost, we obtain a series of subproofs independent from each other. This is the gist of the splitting theorem, and cut-elimination comes as a corollary, by showing that we are free to rearrange these independent subproofs in such a way that the cut is no longer necessary.

We will show that this cut-elimination procedure corresponds to cut-elimination in the non-subatomic system SMLLS. For that, we will pay particular attention to *tame* proofs.

Definition 2.1. We say that an interpretable derivation ϕ in SA is *tame* if the only instances of rules in the scope of an atom are equality rules.

Note that the composition of tame derivations by any relation that is not an atom yields a tame derivation.

Example 2.2. The derivation

$$a{\downarrow}\,\frac{(\perp \parr 1)\ a\ (\perp \parr 1)}{(\perp\ a\ \perp) \parr (1\ a\ 1)}\ a\ \perp$$

in SAMLLS is interpretable but is not tame.

The derivation

$$=\frac{1}{(\perp \parr 1)}\ a\ \perp$$

is tame.

Every proof in SMLLS corresponds to a tame proof in SAMLLS since every rule of SMLLS corresponds to a tame derivation in SAMLLS. This is trivial for every rule, except for the atomic introduction and cut rules

$$\frac{1}{a \parr \bar{a}} \qquad \text{and} \qquad \frac{a \otimes \bar{a}}{\bot} \quad .$$

The introduction rule corresponds to the tame derivation

$$a{\downarrow}\,\frac{=\,\dfrac{=\,\dfrac{1}{1\;a\;1}}{(\bot \parr 1)\;a\;(1 \parr \bot)}}{(\bot\;a\;1) \parr (1\;a\;\bot)} \qquad ,$$

and dually the cut rule corresponds to a tame derivation as well.

Tameness is preserved by splitting and therefore it is preserved by the cut-elimination procedure. The cut-free proofs obtained from proofs in the 'original' system will therefore be tame and correspond to cut-free proofs in SMLLS.

In what follows we will present the splitting theorem for $\mathsf{SAMLLS}^{\downarrow}$. The form of the statement follows the standard scheme for splitting theorems, stemming from the original proof in [26]: it is therefore divided in two results for ease of reading, called *shallow splitting* and *context reduction*. Guided from the generalisation we present in Section 2.2, we use a simple induction measure. We will work modulo associativity, commutativity and unit of $\parr$.

Notation 2.3. We will abuse notation and refer to a derivation ϕ composed only of equality rules as an *equality*.

Definition 2.4. Given a proof ϕ in $\mathsf{SAMLLS}^{\downarrow}$, we define the *length of* ϕ as the number of inference rules in ϕ different from the equality rules for the associativity, commutativity and unit of $\parr$. We denote it by $|\phi|_{\parr}$.

Definition 2.5. We define $=_{\parr}$ as the equivalence relation on formulae defined by the axioms for the associativity, commutativity and unit of $\parr$.

We define the equivalence relation $=_{\parr}$ on derivations following Definition 1.32.

It is straightforward that if $\phi =_{\parr} \psi$, then $|\phi|_{\parr} = |\psi|_{\parr}$.

Theorem 2.6 (Shallow splitting)**.** *For all formulae A, B, C:*

1. *If there is a proof ϕ of $(A \otimes B) \parr C$ in $\mathsf{SAMLLS}^{\downarrow}$, there exist Q_1, Q_2 and*

$$\begin{array}{c} Q_1 \parr Q_2 \\ \psi \,\Vert \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\Vert \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \,\Vert \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi|_{\parr}$.

Furthermore, if ϕ is tame, then ϕ_1, ϕ_2 and ψ are tame.

2. If there is a proof ϕ of $(A \, a \, B) \parr C$ in $\mathsf{SAMLLS}^{\downarrow}$, there exist Q_1, Q_2 and

$$\begin{array}{c} Q_1 \, a \, Q_2 \\ \psi \Vert \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi|_{\parr}$.

Furthermore, if ϕ is tame, then ϕ_1, ϕ_2 are equalities and ψ is tame.

Proof. Given a proof ϕ of $(A \otimes B) \parr C$ in $\mathsf{SAMLLS}^{\downarrow}$ we reduce it to CoS notation for $=_{\parr}$. We proceed by induction on $|\phi|_{\parr}$.

1. If $|\phi|_{\parr} = 0$, then $(A \otimes B) \parr C =_{\parr} 1$. Then, either:

   – $A =_{\parr} B =_{\parr} 1, C =_{\parr} \perp$ and we take

$$\psi \equiv {}_{=}\dfrac{\perp \parr \perp}{{}_{=_{\parr}}\dfrac{\perp}{C}} \quad , \quad \phi_1 \equiv {}_{=_{\parr}}\dfrac{1}{\boxed{{}_{=_{\parr}}\dfrac{1}{A}} \parr \perp} \quad , \quad \phi_2 \equiv {}_{=_{\parr}}\dfrac{1}{\boxed{{}_{=_{\parr}}\dfrac{1}{B}} \parr \perp} \quad ; \text{or}$$

   – $A = \perp, B = C = 1$ and we take

$$\psi \equiv {}_{=}\dfrac{1 \parr \perp}{{}_{=_{\parr}}\dfrac{1}{C}} \quad , \quad \phi_1 \equiv {}_{=_{\parr}}\dfrac{1}{\boxed{{}_{=_{\parr}}\dfrac{\perp}{A}} \parr 1} \quad , \quad \phi_2 \equiv {}_{=_{\parr}}\dfrac{1}{\boxed{{}_{=_{\parr}}\dfrac{1}{B}} \parr \perp} \quad ; \text{or}$$

   – $B = \perp, A = C = 1$ and we take $Q_1 = \perp, Q_2 = 1$

$$\psi \equiv {}_{=}\dfrac{\perp \parr 1}{{}_{=_{\parr}}\dfrac{1}{C}} \quad , \quad \phi_1 \equiv {}_{=_{\parr}}\dfrac{1}{\boxed{{}_{=_{\parr}}\dfrac{1}{A}} \parr \perp} \quad , \quad \phi_2 \equiv {}_{=_{\parr}}\dfrac{1}{\boxed{{}_{=_{\parr}}\dfrac{\perp}{B}} \parr 1} \quad .$$

   If $|\phi|_{\parr} = n > 0$, inspection of the rules provides us the following possible cases:

   (1) $\phi =_{\parr} \boxed{\begin{array}{c} \phi' \Vert \\ {}_{r}\dfrac{(A' \otimes B) \parr C}{(A \otimes B) \parr C} \end{array}}$;

   (2) $\phi =_{\parr} \boxed{\begin{array}{c} \phi' \Vert \\ {}_{r}\dfrac{(A \otimes B') \parr C}{(A \otimes B) \parr C} \end{array}}$;

(3) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{{}_{r}\dfrac{(A\otimes B)\parr C'}{(A\otimes B)\parr C}}}$;

(4) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{{}_{\otimes\downarrow}\dfrac{((A\parr C_1)\otimes(B\parr C_2))\parr C_3}{(A\otimes B)\parr C_1\parr C_2\parr C_3}}}$ with $C =_{\parr} C_1\parr C_2\parr C_3$;

(5) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{{}_{\otimes\downarrow}\dfrac{(((A\otimes B)\parr C_1)\otimes(C_2\parr C_3))\parr C_4}{(A\otimes B)\parr C_2\parr(C_1\otimes C_3)\parr C_4}}}$ with $C =_{\parr} C_2\parr(C_1\otimes C_3)\parr C_4$;

(6) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{{}_{=}\dfrac{(A_1\otimes(A_2\otimes B))\parr C}{((A_1\otimes A_2)\otimes B)\parr C}}}$;

(7) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{{}_{=}\dfrac{(A\otimes B)\parr C}{(B\otimes A)\parr C}}}$;

(8) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{\dfrac{(((A\otimes B)\parr C_1)\otimes 1)\parr C_2}{(A\otimes B)\parr C_1\parr C_2}}}$ with $C =_{\parr} C_1\parr C_2$;

(9) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{\dfrac{(1\otimes((A\otimes B)\parr C_1))\parr C_2}{(A\otimes B)\parr C_1\parr C_2}}}$ with $C =_{\parr} C_1\parr C_2$;

(10) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{\dfrac{A\parr C}{(A\otimes 1)\parr C}}}$ with $B =_{\parr} 1$;

(11) $\phi =_{\parr} \overset{\phi'\,\Pi}{\boxed{\dfrac{A\parr C}{(1\otimes B)\parr C}}}$ with $A =_{\parr} 1$.

(1) Since $|\phi'|_{\parr} = n-1$, we apply the induction hypothesis to ϕ'. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1\parr Q_2 \\ \psi\,\| \\ C \end{array} \quad , \quad \phi_1 \equiv \overset{\phi_1'\,\Pi}{\boxed{{}_{r}\dfrac{A'}{A}}}\parr Q_1 \quad , \quad \begin{array}{c} \phi_2\,\Pi \\ B\parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} = |\phi_1'|_{\parr} + |\phi_2|_{\parr} + 1 \leq |\phi'|_{\parr} + 1 = |\phi|_{\parr}$.

If ϕ is tame, then ψ, ϕ_1' and ϕ_2 are tame. Furthermore, since ϕ is tame r is tame, and therefore ϕ_1 is interpretable.

(2) This case is analogous to (1).

(3) We apply the induction hypothesis to ϕ'. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1 \parr Q_2 \\ {\scriptstyle \psi'} \, \| \\ \boxed{ {\scriptstyle r}\, \dfrac{C'}{C} } \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \, \| \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \, \| \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then ψ', ϕ_1 and ϕ_2 are tame. Furthermore, since ϕ is tame r is tame. Therefore ψ is tame.

(4) We apply the induction hypothesis to ϕ'. There exist Q_1', Q_2' and

$$\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi'} \, \| \\ C_3 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \, \| \\ A \parr C_1 \parr Q_1' \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \, \| \\ B \parr C_2 \parr Q_2' \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then ψ', ϕ_1 and ϕ_2 are tame.

We take $Q_1 = C_1 \parr Q_1'$, $Q_2 = C_2 \parr Q_2'$ and we have

$$\psi \equiv \quad \begin{array}{c} C_1 \parr Q_1' \parr C_2 \parr Q_2' \\ {\scriptstyle =_{\parr}} \overline{\overline{\hspace{6cm}}} \\ C_1 \parr C_2 \parr \boxed{\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi'} \, \| \\ C_3 \end{array}} \\ {\scriptstyle =_{\parr}} \overline{\overline{\hspace{6cm}}} \\ C \end{array} \quad .$$

If ϕ is tame, since ϕ_1 and ϕ_2 are tame, C_1, Q_1' and C_2, Q_2' are interpretable. Then, since ψ' is tame, ψ is tame.

(5) We apply the induction hypothesis to ϕ'. There exist Q_1', Q_2' and

$$\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi_1} \, \| \\ C_4 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1'} \, \| \\ (A \otimes B) \parr C_1 \parr Q_1' \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2'} \, \| \\ C_2 \parr C_3 \parr Q_2' \end{array}$$

such that $|\phi_1'|_{\parr} + |\phi_2'|_{\parr} \leq |\phi'|_{\parr}$.

We apply the induction hypothesis to ϕ_1'. There exist Q_1, Q_2 and

$$\psi \equiv \begin{array}{c} \dfrac{\boxed{\begin{array}{c} Q_1 \parr Q_2 \\ {\scriptstyle \psi_2} \Vert \\ C_1 \parr Q_1' \end{array}} \otimes \boxed{\begin{array}{c} {\scriptstyle \phi_2'} \Pi \\ C_2 \parr C_3 \parr Q_2' \end{array}}}{(C_1 \otimes C_3) \parr C_2 \parr \boxed{\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi_1} \Vert \\ C_4 \end{array}}} \otimes\!\downarrow \\ \hline\hline C \quad =_{\parr} \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \Pi \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \Pi \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi_1'|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then ψ_1, ϕ_1' and ϕ_2' are tame. Therefore, ψ_2, ϕ_1 and ϕ_2 are tame and thus ψ is tame.

(6) We apply the induction hypothesis to ϕ'. There exist Q_1', Q_2' and

$$\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi_1} \Vert \\ C \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1'} \Pi \\ A_1 \parr Q_1' \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2'} \Pi \\ (A_2 \otimes B) \parr Q_2' \end{array}$$

such that $|\phi_1'|_{\parr} + |\phi_2'|_{\parr} \leq |\phi'|_{\parr}$.

We apply the induction hypothesis to ϕ_2'. There exist M, Q_2 and

$$\begin{array}{c} M \parr Q_2 \\ {\scriptstyle \psi_2} \Vert \\ Q_2' \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \zeta} \Pi \\ A_2 \parr M \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \Pi \\ B \parr Q_2 \end{array}$$

such that $|\zeta|_{\parr} + |\phi_2|_{\parr} \leq |\phi_2'|_{\parr}$.

We take $Q_1 \equiv Q_1' \parr M$ and

$$\psi \equiv \begin{array}{c} (Q_1' \parr M) \parr Q_2 \\ \hline\hline Q_1' \parr \boxed{\begin{array}{c} M \parr Q_2 \\ {\scriptstyle \psi_2} \Vert \\ Q_2' \end{array}} \quad =_{\parr} \\ {\scriptstyle \psi_1} \Vert \\ C \end{array} \quad , \quad \phi_1 \equiv \otimes\!\downarrow \dfrac{\boxed{\begin{array}{c} {\scriptstyle \phi_1'} \Pi \\ A_1 \parr Q_1' \end{array}} \otimes \boxed{\begin{array}{c} {\scriptstyle \zeta} \Pi \\ A_2 \parr M \end{array}}}{(A_1 \otimes A_2) \parr (Q_1' \parr M)} \quad .$$

We have:
$|\phi_1|_{\parr} + |\phi_2|_{\parr} = |\phi_1'|_{\parr} + |\zeta|_{\parr} + 1 + |\phi_2|_{\parr} \leq |\phi_1'|_{\parr} + |\phi_2'|_{\parr} + 1 \leq |\phi'|_{\parr} + 1 = |\phi|_{\parr}$.

If ϕ is tame, ψ_1, ϕ_1' and ϕ_2' are tame. Then, ψ_2, ζ and ϕ_2 are tame. Therefore, ψ and ϕ_1 are tame.

(7) We apply the induction hypothesis to ϕ'. There are Q_1', Q_2' and

$$\begin{array}{c} Q_2 \parr Q_1 \\ {\scriptstyle \psi'} \, \| \\ C \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \, \Pi \\ B \parr Q_2 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \, \Pi \\ A \parr Q_1 \end{array}$$

such that $|\phi_1|_\parr + |\phi_2|_\parr \leq |\phi|_\parr$.

We take

$$\psi \equiv \begin{array}{c} {\scriptstyle \Rightarrow_\parr} \dfrac{Q_1 \parr Q_2}{Q_1 \parr Q_1} \\ {\scriptstyle \psi'} \, \| \\ C \end{array} \quad .$$

If ϕ is tame, ψ', ϕ_1 and ϕ_2 are tame, and thus ψ is tame as well.

(8) We apply the induction hypothesis to ϕ'. There exist Q_1', Q_2' and

$$\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi_1} \, \| \\ C_2 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1'} \, \Pi \\ (A \otimes B) \parr C_1 \parr Q_1' \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2'} \, \Pi \\ 1 \parr Q_2' \end{array}$$

such that $|\phi_1'|_\parr + |\phi_2'|_\parr \leq |\phi'|_\parr$.

We apply the induction hypothesis to ϕ_1'. There exist Q_1, Q_2 and

$$\psi \equiv {\scriptstyle \Rightarrow_\parr} \dfrac{{\scriptstyle \otimes\downarrow} \dfrac{\boxed{\begin{array}{c} Q_1 \parr Q_2 \\ {\scriptstyle \psi_2} \, \| \\ C_1 \parr Q_1' \end{array}} \otimes \boxed{\begin{array}{c} {\scriptstyle \phi_2'} \, \Pi \\ 1 \parr Q_2' \end{array}}}{(C_1 \otimes 1) \parr \boxed{\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi_1} \, \| \\ C_2 \end{array}}}}{C} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \, \Pi \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \, \Pi \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_\parr + |\phi_2|_\parr \leq |\phi_1'|_\parr \leq |\phi'|_\parr \leq |\phi|_\parr$.

If ϕ is tame, then so are ψ_1, ϕ_1' and ϕ_2'. Therefore, ψ_2, ϕ_1 and ϕ_2 are tame, and so is ψ.

(9) This case is analogous to case (8).

(10) We take

$$\psi \equiv {\scriptstyle \Rightarrow_\parr} \dfrac{C \parr \bot}{C} \quad , \quad \phi_1 \equiv \begin{array}{c} {\scriptstyle \phi'} \, \Pi \\ A \parr C \end{array} \quad , \quad \phi_2 \equiv {\scriptstyle \Rightarrow_\parr} \dfrac{1}{\boxed{{\scriptstyle \Rightarrow_\parr} \dfrac{1}{B}} \parr \bot} \quad .$$

We have $|\phi_1|_\parr + |\phi_2|_\parr \leq |\phi|_\parr$.

If ϕ is tame, then C is interpretable and ϕ' is tame and thus ψ and ϕ_1 are tame. ψ_2 is tame.

(11) This case is analogous to case (10).

2. If $|\phi|_{\parr} = 0$, then either

– $A =_{\parr} B =_{\parr} 1, C =_{\parr} \perp$ and we take

$$\psi \equiv {}_{=}\frac{\perp\, a \perp}{{}_{=_{\parr}}\dfrac{\perp}{\overline{\overline{C}}}} \;, \qquad \phi_1 \equiv {}_{=_{\parr}}\frac{1}{\boxed{{}_{=_{\parr}}\dfrac{1}{\overline{\overline{A}}}} \parr \perp} \;, \qquad \phi_2 \equiv {}_{=_{\parr}}\frac{1}{\boxed{{}_{=_{\parr}}\dfrac{1}{\overline{\overline{B}}}} \parr \perp} \;,$$

with $|\phi_1|_{\parr} = |\phi_2|_{\parr} = 0$;

– or $A =_{\parr} B =_{\parr} \perp, C =_{\parr} 1$ and we take

$${}_{=}\frac{1\, a\, 1}{{}_{=_{\parr}}\dfrac{1}{\overline{\overline{C}}}} \;, \qquad \phi_1 \equiv {}_{=_{\parr}}\frac{1}{\boxed{{}_{=_{\parr}}\dfrac{\perp}{\overline{\overline{A}}}} \parr 1} \;, \qquad \phi_2 \equiv {}_{=_{\parr}}\frac{1}{\boxed{{}_{=_{\parr}}\dfrac{\perp}{\overline{\overline{B}}}} \parr 1} \;.$$

with $|\phi_1|_{\parr} = |\phi_2|_{\parr} = 0$.

If $|\phi|_{\parr} = n > 0$ and $A\, a\, B \neq_{\parr} u$, inspection of the rules provides us the following possible cases:

(1) $\phi =_{\parr} \boxed{{}_{r}\dfrac{\overset{\phi' \,\Vert}{(A'\, a\, B) \parr C}}{(A\, a\, B) \parr C}}$;

(2) $\phi =_{\parr} \boxed{{}_{r}\dfrac{\overset{\phi' \,\Vert}{(A\, a\, B') \parr C}}{(A\, a\, B) \parr C}}$;

(3) $\phi =_{\parr} \boxed{{}_{r}\dfrac{\overset{\phi' \,\Vert}{(A\, a\, B) \parr C'}}{(A\, a\, B) \parr C}}$;

(4) $\phi =_{\parr} \boxed{{}_{\otimes\downarrow}\dfrac{\overset{\phi' \,\Vert}{((A \parr C_1)\, a\, (B \parr C_2)) \parr C_3}}{(A\, a\, B) \parr (C_1\, a\, C_2) \parr C_3}}$ with $C =_{\parr} (C_1\, a\, C_2) \parr C_3$;

(5) $$\phi =_{\parr} \begin{array}{c} \phi' \Vert \\ \boxed{{\scriptstyle \otimes\downarrow}\dfrac{(((A\,a\,B)\parr C_1)\otimes(C_2\parr C_3))\parr C_4}{(A\,a\,B)\parr C_2\parr(C_1\otimes C_3)\parr C_4}} \end{array} \qquad \text{with } C =_{\parr} C_2\parr(C_1\otimes C_3)\parr C_4\ ;$$

(6) $$\phi =_{\parr} \begin{array}{c} \phi' \Vert \\ \boxed{{\scriptstyle =}\dfrac{(((A\,a\,B)\parr C_1)\otimes 1)\parr C_2}{(A\,a\,B)\parr C_1\parr C_2}} \end{array} \qquad \text{with } C =_{\parr} C_1\parr C_2\ ;$$

(7) $$\phi =_{\parr} \begin{array}{c} \phi' \Vert \\ \boxed{{\scriptstyle =}\dfrac{(1\otimes((A\,a\,B)\parr C_1))\parr C_2}{(A\,a\,B)\parr C_1\parr C_2}} \end{array} \qquad \text{with } C =_{\parr} C_1\parr C_2\ ;$$

(8) $$\phi =_{\parr} \begin{array}{c} \phi' \Vert \\ \boxed{{\scriptstyle =}\dfrac{1\parr C}{(1\,a\,1)\parr C}} \end{array} \qquad \text{with } A =_{\parr} B =_{\parr} 1\ ;$$

(9) $$\phi =_{\parr} \begin{array}{c} \phi' \Vert \\ \boxed{{\scriptstyle =}\dfrac{\bot\parr C}{(\bot\,a\,\bot)\parr C}} \end{array} \qquad \text{with } A =_{\parr} B =_{\parr} 1\ .$$

(1) We apply the induction hypothesis to ϕ'. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1\,a\,Q_2 \\ \Vert \\ C \end{array} \quad , \quad \phi_1 \equiv \begin{array}{c} \phi_1' \Vert \\ \boxed{{\scriptstyle r}\dfrac{A'}{A}}\parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ B\parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} = |\phi_1'|_{\parr} + 1 + |\phi_2|_{\parr} \leq |\phi'|_{\parr} + 1 = |\phi|_{\parr}$.

If ϕ is tame, ψ is tame and ϕ_1' and ϕ_2 are equalities. r is an equality, and therefore ϕ_1 is an equality.

(2) This case is analogous to (1).

(3) We apply the induction hypothesis to ϕ'. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1\,a\,Q_2 \\ \psi' \Vert \\ \boxed{{\scriptstyle r}\dfrac{C'}{C}} \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \\ A\parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ B\parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, so are ψ' and r and thus so is ψ. ϕ_1 and ϕ_2 are equalities.

(4) We apply the induction hypothesis to ϕ'. There exist Q'_1, Q'_2 and

$$\begin{array}{c} Q'_1 \; a \; Q'_2 \\ \psi' \Vert \\ C_3 \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \\ A \parr C_1 \parr Q'_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ B \parr C_2 \parr Q'_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

We take $Q_1 = C_1 \parr Q'_1$, $Q_2 = C_2 \parr Q'_2$ and

$$\psi \equiv \quad {=_{\parr}}\dfrac{a{\downarrow}\dfrac{(C_1 \parr Q'_1) \; a \; (C_2 \parr Q'_2)}{(C_1 \; a \; C_2) \parr \boxed{\begin{array}{c} Q'_1 \; a \; Q'_2 \\ \psi' \Vert \\ C_3 \end{array}}}}{C} \quad .$$

If ϕ is tame, then ψ' is tame and ϕ_1 and ϕ_2 are equalities. Then $C_1 \parr Q'_1 = 1$ or $C_1 \parr Q'_1 = \bot$ and $C_2 \parr Q'_2 = 1$ or $C_2 \parr Q'_2 = \bot$. Therefore, $(C_1 \parr Q'_1) \, a \, (C_2 \parr Q'_2)$ and $C_1 \; a \; C_2$ are interpretable and ψ is tame.

(5) We apply the induction hypothesis to ϕ'. There exist Q'_1, Q'_2 and

$$\begin{array}{c} Q'_1 \parr Q'_2 \\ \psi_1 \Vert \\ C_4 \end{array} \quad , \quad \begin{array}{c} \phi'_1 \Vert \\ (A \; a \; B) \parr C_1 \parr Q'_1 \end{array} \quad , \quad \begin{array}{c} \phi'_2 \Vert \\ C_2 \parr C_3 \parr Q'_2 \end{array}$$

such that $|\phi'_1|_{\parr} + |\phi'_2|_{\parr} \leq |\phi'|_{\parr}$.

We apply the induction hypothesis to ϕ'_1. There exist Q_1, Q_2 and

$$\psi \equiv \quad {=_{\parr}}\dfrac{\otimes{\downarrow}\dfrac{\boxed{\begin{array}{c} Q_1 \; a \; Q_2 \\ \psi_2 \Vert \\ C_1 \parr Q'_1 \end{array}} \otimes \boxed{\begin{array}{c} \phi'_2 \Vert \\ C_2 \parr C_3 \parr Q'_2 \end{array}}}{(C_1 \otimes C_3) \parr C_2 \parr \boxed{\begin{array}{c} Q'_1 \parr Q'_2 \\ \psi_1 \Vert \\ C_4 \end{array}}}}{C} \quad , \quad \begin{array}{c} \phi_1 \Vert \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi'_1|_{\parr} \leq |\phi'|_{\parr} \leq |\phi|_{\parr}$.

If ϕ is tame, then ψ_1, ϕ'_1, ϕ'_2 and ψ_2 are tame. Therefore ψ is tame. Furthermore, by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(6) We apply the induction hypothesis to ϕ'. There exist Q_1', Q_2' and

$$\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi_1} \Vert \\ C_2 \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \phi_1'} \Pi \\ (A \mathbin{a} B) \parr C_1 \parr Q_1' \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2'} \Pi \\ 1 \parr Q_2' \end{array}$$

such that $|\phi_1'|_\parr + |\phi_2'|_\parr \leq |\phi'|_\parr$.

We apply the induction hypothesis to ϕ_1'. There exist Q_1, Q_2 and

$$\psi \equiv \begin{array}{l} \otimes\!\downarrow \dfrac{\boxed{\begin{array}{c} Q_1 \mathbin{a} Q_2 \\ {\scriptstyle \psi_2} \Vert \\ C_1 \parr Q_1' \end{array}} \otimes \boxed{\begin{array}{c} {\scriptstyle \phi_2'} \Pi \\ 1 \parr Q_2' \end{array}}}{(C_1 \otimes 1) \parr \boxed{\begin{array}{c} Q_1' \parr Q_2' \\ {\scriptstyle \psi_1} \Vert \\ C_2 \end{array}}} \\ =_\parr \overline{\overline{\qquad\qquad C \qquad\qquad}} \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \phi_1} \Pi \\ A \parr Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2} \Pi \\ B \parr Q_2 \end{array}$$

such that $|\phi_1|_\parr, |\phi_2|_\parr \leq |\phi_1'|_\parr \leq |\phi'|_\parr \leq |\phi|_\parr$.

If ϕ is tame, then ψ_1, ϕ_1', ϕ_2' and ψ_2 are tame. Therefore ψ is tame. Furthermore, by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(7) This case is analogous to case (5).

(8) We take

$$\phi_1 \equiv =_\parr \dfrac{1}{\boxed{=_\parr \dfrac{1}{A}} \parr \bot} \quad , \qquad \phi_2 \equiv =_\parr \dfrac{1}{\boxed{=_\parr \dfrac{1}{B}} \parr \bot} \qquad \text{and}$$

$$\psi \equiv \otimes\!\downarrow \dfrac{\left(= \dfrac{\bot \mathbin{a} \bot}{\bot} \parr \bot \right) \otimes \boxed{\begin{array}{c} {\scriptstyle \phi} \Pi \\ 1 \parr C \end{array}}}{\boxed{= \dfrac{\bot \otimes 1}{\bot}} \parr \bot \parr C} \quad ,$$

with $|\phi_1|_\parr = |\phi_2|_\parr = 0$.

If ϕ is tame, ψ is tame. Furthermore, ϕ_1 and ϕ_2 are equalities.

(9) We take

$$\phi_1 \equiv =_\parr \dfrac{1}{\boxed{=_\parr \dfrac{\bot}{A}} \parr 1} \quad , \qquad \phi_2 \equiv =_\parr \dfrac{1}{\boxed{=_\parr \dfrac{\bot}{B}} \parr 1} \qquad \text{and}$$

$$\psi \equiv \otimes\!\!\downarrow \frac{\left(= \dfrac{1 \; a \; 1}{1} \parr \bot \right) \otimes \boxed{\begin{array}{c} \phi \Vert \\ \bot \parr C \end{array}}}{\boxed{= \dfrac{1 \otimes \bot}{\bot}} \parr \bot \parr C} \quad ,$$

with $|\phi_1|_{\parr} = |\phi_2|_{\parr} = 0$.

If ϕ is tame, ψ is tame. Furthermore, ϕ_1 and ϕ_2 are equalities.

□

Note the big similarities in the case analysis for both clauses of the theorem. In fact, in the general splitting theorem we will provide a case analysis that holds for every connective.

To grasp the generalization, it is important to note that the base cases rely on the dualities in the equational theory. If A and B are equal to constants v and w respectively, there need to be dual constants $\bar{v}$ and $\bar{w}$ such that $v \parr \bar{v} = 1$ and $w \parr \bar{w} = 1$. Furthermore, tameness is preserved by splitting because of some properties of the interpretation map, most importantly those that allow us to guarantee the interpretability of the premiss in case 2.(4). These will be fundamental requirements for the generalised splitting theorem.

Shallow splitting tells us that from 'shallow' contexts where the main connective is $\parr$ we can follow occurrences of $\otimes$ and of the atoms up in the proof and obtain independent subproofs. We can now apply shallow splitting starting from the outermost occurrences of $\otimes$ or the atoms, and apply this process recursively on every subproof to obtain a series of nested subproofs that in a way make-up the original proof. We formalise this recursive process in the following theorem.

Definition 2.7. We say that a context $H\{\ \}$ is *provable* if $H\{1\} = 1$.

Definition 2.8. Given a context $S\{\ \}$ we define its *height* as the number of instances of $\otimes$ and a that $\{\ \}$ is in the scope of. We denote it by $|S|_{\parr}$.

Example 2.9. The height of $S\{\ \} = (\bot \; a \; (1 \otimes \{\ \})) \parr (1 \; a \; \bot)$ is 2.

Theorem 2.10 (Context Reduction)**.** *For any formula A and any context S, given a proof* $\begin{array}{c} \phi \Vert \mathsf{SAMLLS}{\downarrow} \\ S\{A\} \end{array}$ *there exist a provable context $H\{\ \}$, a formula K and derivations*

$$\begin{array}{c} \zeta \Big\Vert \mathsf{SAMLLS}{\downarrow} \\ K \parr A \end{array} \quad , \qquad \begin{array}{c} H\{K \parr \{\ \}\} \\ \chi \Vert \\ S\{\ \} \end{array} \quad ,$$

such that if ϕ is tame, then ζ is tame.

Furthermore, if $\{\ \}$ is not in the scope of an atom in $S\{\ \}$ and ϕ is tame, then χ is tame.

Proof. We proceed by induction on $|S|_{\parr}$.

- If $S\{A\} =_{\parr} A \parr K$, it is clear.

- If $S\{A\} =_{\parr} (S'\{A\} \otimes L) \parr M$, we apply Theorem 2.6. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1 \parr Q_2 \\ \psi \Vert \\ M \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \\ S'\{A\} \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ L \parr Q_2 \end{array} \quad .$$

We apply the induction hypothesis to $S'\{A\} \parr Q_1$. There exist a provable context $H\{\ \}$, a formula K and derivations

$$\begin{array}{c} \zeta \Big\Vert \mathsf{SAMLLS}^{\downarrow} \\ K \parr A \end{array} \quad , \quad \chi \equiv \otimes\!\!\downarrow \frac{\boxed{\begin{array}{c} H\{K \parr \{\ \}\} \\ \chi' \Vert \\ S'\{\ \} \parr Q_1 \end{array}} \otimes \boxed{\begin{array}{c} \phi_2 \Vert \\ L \parr Q_2 \end{array}}}{(S'\{\ \} \otimes L) \parr \boxed{\begin{array}{c} Q_1 \parr Q_2 \\ \psi \Vert \\ M \end{array}}} \quad .$$

We take $H\{\ \} \equiv H'\{\ \} \otimes 1$.

If ϕ is tame, then ζ is tame. If $\{\ \}$ is not in the scope of an atom in $S\{\ \}$ and ϕ is tame, then χ' is tame. Furthermore, ϕ_2 and ψ are tame, and therefore χ is tame.

- If $S\{A\} =_{\parr} (S'\{A\}\ a\ L) \parr M$, we apply Theorem 2.6. There exist Q_1, Q_2 and

$$\begin{array}{c} Q_1\ a\ Q_2 \\ \psi \Vert \\ M \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \\ S'\{A\} \parr Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \Vert \\ L \parr Q_2 \end{array} \quad .$$

We apply the induction hypothesis to $S'\{A\} \parr Q_1$. There exist a provable context H', a formula K and derivations

$$\begin{array}{c} \zeta \Big\Vert \mathsf{SAMLLS}^{\downarrow} \\ K \parr A \end{array} \quad , \quad \chi \equiv a\!\downarrow \frac{\boxed{\begin{array}{c} H'\{K \parr \{\ \}\} \\ \chi' \Vert \\ S'\{\ \} \parr Q_1 \end{array}}\ a\ \boxed{\begin{array}{c} \phi_2 \Vert \\ L \parr Q_2 \end{array}}}{(S'\{\ \} \otimes L) \parr \boxed{\begin{array}{c} Q_1\ a\ Q_2 \\ \psi \Vert \\ M \end{array}}} \quad .$$

We take $H\{\ \} \equiv H'\{\ \}\ a\ 1$.

If ϕ is tame, then ζ is tame.

$\square$

The splitting results are stronger than cut-elimination: they give us information about the structure of a proof and the 'pieces' from which it's built. Cut-elimination is a corollary of these results, stemming from our ability to rearrange these pieces in a way that suits us and still obtain a proof.

To show that the cut is admissible in a proof we will follow the relations a and $\otimes$ that take part in the cut to find what independent subproofs they belong to. We will then rearrange them in such a way that the cut is no longer needed.

For example, we consider the following simple proof:

$$\otimes\!\downarrow \frac{\boxed{a\!\downarrow \dfrac{\boxed{1 \parr \bot}\; a\; \boxed{\bot \parr 1}}{(1\; a\; \bot) \parr (\bot\; a\; 1)}} \otimes \boxed{a\!\downarrow \dfrac{\boxed{\bot \parr 1}\; a\; \boxed{1 \parr \bot}}{(\bot\; a\; 1) \parr (1\; a\; \bot)}}}{a\!\uparrow \dfrac{(1\; a\; \bot) \otimes (\bot\; a\; 1)}{(1 \otimes \bot)\; a\; (\bot \otimes 1)} \parr (\bot\; a\; 1) \parr (1\; a\; \bot)} \quad .$$

We follow the relations participating in the cut (in red) to find the $\boxed{\text{boxed}}$ independent subproofs via context reduction and splitting. We can then rearrange them to obtain the following cut-free proof:

$$a\!\downarrow \frac{\boxed{\otimes\!\downarrow \dfrac{\boxed{1 \parr \bot} \otimes \boxed{\bot \parr 1}}{(1 \otimes \bot) \parr (\bot \parr 1)}}\; a\; \boxed{\otimes\!\downarrow \dfrac{\boxed{\bot \parr 1} \otimes \boxed{1 \parr \bot}}{(\bot \otimes 1) \parr 1 \parr \bot}}}{(1 \otimes \bot)\; a\; (\bot \otimes 1) \parr \boxed{a\!\downarrow \dfrac{(\bot \parr 1)\; a\; (1 \parr \bot)}{(\bot\; a\; 1) \parr (1\; a\; \bot)}}} \quad .$$

Through the following corollary we will show that such a rearrangement is always possible, and therefore the cut is admissible.

Corollary 2.11 (Cut Elimination)**.** *For any formulae A, B, C, D, any context S and any proof*

$$\phi \equiv \begin{array}{c} \Vert\, \mathsf{SAMLLS}^{\downarrow} \\ S\left\{ a\!\uparrow \dfrac{(A\; a\; B) \otimes (C\; a\; D)}{(A \otimes C)\; a\; (B \otimes D)} \right\} \end{array} \quad ,$$

there is a proof

$$\begin{array}{c} \phi' \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ S\{(A \otimes C)\; a\; (B \otimes D)\} \end{array} \quad .$$

Furthermore, if ϕ is tame then ϕ' is tame.

Proof. Given a proof $\begin{array}{c} \Vert\, \mathsf{SAMLLS}^{\downarrow} \\ S\{(A\; a\; B) \otimes (C\; a\; D)\} \end{array}$, we apply Theorem 2.10.

There exist a provable context H, a formula K and derivations

$$\begin{array}{c} \zeta \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ K \parr ((A \; a \; B) \otimes (C \; a \; D)) \end{array} \quad , \qquad \begin{array}{c} H\{K \parr \{\ \}\} \\ \chi \,\Vert \\ S\{\ \} \end{array} \quad .$$

We apply Theorem 2.6 to ζ. There are formulae Q_1, Q_2 and derivations

$$\begin{array}{c} Q_1 \parr Q_2 \\ \psi \,\Vert \\ K \end{array} \quad , \qquad \begin{array}{c} \phi_1 \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ (A \; a \; B) \parr Q_1 \end{array} \quad , \qquad \begin{array}{c} \phi_2 \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ (C \; a \; D) \parr Q_2 \end{array} \quad .$$

We apply Theorem 2.6 to ϕ_1. There are formulae Q_A, Q_B and derivations

$$\begin{array}{c} Q_A \; a \; Q_B \\ \psi_1 \,\Vert \\ Q_1 \end{array} \quad , \qquad \begin{array}{c} \phi_A \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ A \parr Q_A \end{array} \quad , \qquad \begin{array}{c} \phi_B \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ B \parr Q_B \end{array} \quad .$$

We apply Theorem 2.6 to ϕ_2. There are formulae Q_C, Q_D and derivations

$$\begin{array}{c} Q_C \; a \; Q_D \\ \psi_2 \,\Vert \\ Q_2 \end{array} \quad , \qquad \begin{array}{c} \phi_C \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ C \parr Q_C \end{array} \quad , \qquad \begin{array}{c} \phi_D \,\Vert\, \mathsf{SAMLLS}^{\downarrow} \\ D \parr Q_D \end{array} \quad .$$

Finally then, there exists a proof in $\mathsf{SAMLLS}^{\downarrow}$:

$$\phi' = \begin{array}{c} H \left\{ a{\downarrow}\dfrac{\boxed{\otimes{\downarrow}\dfrac{\boxed{\begin{array}{c}\phi_A \Vert \\ A \parr Q_A\end{array}} \otimes \boxed{\begin{array}{c}\phi_C \Vert \\ C \parr Q_C\end{array}}}{(A \otimes C) \parr Q_A \parr Q_C}} \; a \; \boxed{\otimes{\downarrow}\dfrac{\boxed{\begin{array}{c}\phi_B \Vert \\ B \parr Q_B\end{array}} \otimes \boxed{\begin{array}{c}\phi_D \Vert \\ D \parr Q_D\end{array}}}{(B \otimes D) \parr Q_B \parr Q_D}}}{((A \otimes C) \; a \; (B \otimes D)) \parr \boxed{\begin{array}{c} \boxed{a{\downarrow}\dfrac{(Q_A \parr Q_C) \; a \; (Q_B \parr Q_D)}{\boxed{\begin{array}{c} Q_A \; a \; Q_B \\ \psi_1 \Vert \\ Q_1 \end{array}} \parr \boxed{\begin{array}{c} Q_C \; a \; Q_D \\ \psi_2 \Vert \\ Q_2 \end{array}}}} \\ \psi \,\Vert \\ K \end{array}}} \right\} \\ \chi \,\Vert \\ S\{(Q_A \parr Q_C) \; a \; (Q_B \parr Q_D)\} \end{array} \quad .$$

If ϕ is tame, then $\{\ \}$ is not in the scope of an atom in $S\{\ \}$. Then ζ and χ are tame. ψ_1, ψ_2, ψ_3 are tame as well. ϕ_1 and ϕ_2 are equalities. Furthermore, since $(A \otimes C) \, a \, (B \otimes D)$ is interpretable, then $(A \otimes C)$ and $(B \otimes C)$ are of the form $1 \otimes 1$ or $\bot \otimes 1$. Therefore, the instances of $\otimes{\downarrow}$ are trivially of the form $\dfrac{1}{1}$ and can be replaced by equalities. ϕ' is then tame.

□

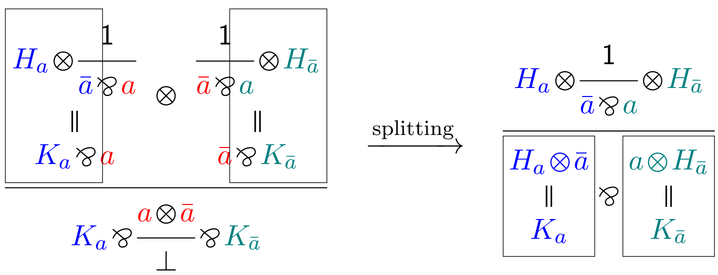
Note that in this last proof we have implicitly made use of the associativity and commutativity of $\parr$. In fact this will be a requirement in the generalised splitting theorem.

Since every proof of SMLLS corresponds to a tame proof in SAMLLS, the cut-free proof obtained from it will be tame and therefore interpretable. This cut-elimination procedure therefore corresponds to cut-elimination in SMLLS.

It is interesting to observe that at no point in the reasoning leading us to cut-elimination have we required formulae to be interpretable. Splitting and the admissibility of up-rules hold for the full subatomic language, and in particular for interpretable proofs.

2.2 General splitting

Splitting is based on a simple idea: to show that an atomic cut involving a and $\bar{a}$ is admissible, we follow a and $\bar{a}$ to the top of the derivation to find two independent subderivations, the premisses of which contain the dual of a and the dual of $\bar{a}$ respectively. In this way we obtain two proofs that don't interact above the cut, that we can rearrange to get a new cut-free proof.



Proofs of cut-elimination by splitting therefore rely on two main properties of a proof system: the *dualities* present in it to ensure that each of the independent subproofs contains the dual of an atom involved in the cut, and the *shape* of the linear rules ensuring that the two proofs remain independent above the cut. It is precisely a formal characterisation of these properties that we will provide, enabling us to understand *why* they are enough to guarantee cut-elimination. We therefore show how the interaction of linear rules and the cut affects cut-elimination.

Since the splitting proof consists on being able to follow relations through a proof to obtain the subproofs that compose it, its generalisation will be based on a characterisation of the relations that we can follow in such a way. In a system with only these relations, cut-elimination will be a mere corollary of splitting as is the case in $\mathsf{SAMLLS}^{\downarrow}$.

To follow a relation through the proof from the bottom to the top, we require their scope to widen. As we observed in $\mathsf{SAMLLS}^{\downarrow}$, the scope of $\otimes$ and a in the inference rules only widens when reading bottom-up. Accordingly, we will consider systems where the shape of the rules ensures the widening of the scope.

Notation 2.12. In what follows we will consider a subatomic system $\mathsf{SA}^{\downarrow}$ with set of formulae $\mathscr{F}$, set of relations $\mathscr{R}$, set of constants $\mathscr{U}$ and a natural interpretation I whose inference rules are *all down-rules*.

A proof in SA is a derivation with premiss $1 \in \mathscr{U}$.

Definition 2.13. We say that a relation α is *contractive* in $\mathsf{SA}^{\downarrow}$ if there is an inference rule

$$\frac{(A\ \alpha\ B)\ \ \nu\ \ (C\ \alpha\ \ D)}{(A\ \nu\ C)\ \ \alpha\ \ (B\ \nu^m\ D)}\quad \text{for some } \nu \in \mathscr{R}$$

in $\mathsf{SA}^{\downarrow}$.

Otherwise, we say that the relation α is *non-contractive*.

Example 2.14. In $\mathsf{SAMLLS}^{\downarrow}$ (Figure 2-1), $\otimes$ and a are non-contractive.

Example 2.15. In SAKS (Figure 1-1), a is contractive since in the rule ac its scope shrinks from bottom to top. Likewise, $\wedge$ is contractive.

In $\mathsf{SAMLLS}^{\downarrow}$ the only contractive relation is $\invamp$. The property distinguishing $\invamp$ from a and $\otimes$ is in fact that it is the *minimal* relation: it is the relation that appears in the excluded middle rules that introduce the dualities. In particular, the fact that $u \invamp \bar{u} = 1$, for every constant u is fundamental to prove the base cases of Theorem 2.6. In every propositional system with an identity rule that introduces dualities there is such a distinguished relation. We will characterise *splittable systems*, i.e., systems with sufficient conditions to ensure cut-elimination through a splitting procedure.

In splittable systems, mimicking the case of MLL, we will require that all relations except for a distinguished relation $+$ be non-contractive so that we are able to follow them in a proof, and that there be a rule $u + \bar{u} = 1$ for every constant u.

Furthermore, when looking for the nested subproofs provided by context reduction in Theorem 2.10, we start from the outermost occurrence of a or $\otimes$ in the conclusion of a proof, and apply shallow splitting recursively. To piece together all the subproofs in such a way that we obtain a provable context, we can see that a fundamental property of a and $\otimes$ is that $1\,a\,1 = 1$ and $1 \otimes 1 = 1$. In splittable systems we will follow the same procedure, and will therefore require that $1\ \alpha^M\ 1 = 1$ for every α.

Lastly, we implicitly made use of the associativity and commutativity of $\invamp$. We will in the same way require associativity and commutativity of $+$.

Definition 2.16. A system $\mathsf{SA}^{\downarrow}$ is *splittable* if:

1. There is a strong relation $\times$ with unit 1 and dual $+$ with unit 0,
2. Every relation $\alpha \neq +$ is non-contractive,
3. There is a constant assignment $u + \bar{u} = 1$ for every unit $u \in \mathscr{U}$,
4. $+$ is associative and commutative,
5. $1\ \alpha^M\ 1 = 1$ for every α.

$$
a\!\downarrow \frac{(A \vee B)\; a\; (C \vee D)}{(A\; a\; C) \vee (B\; a\; D)}
$$

$$
\wedge\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)}
$$

Figure 2-2: $\mathsf{SAKS}^{\downarrow}$

$$
a\!\downarrow \frac{(A \parr B)\; a\; (C \parr D)}{(A\; a\; C) \parr (B\; a\; D)}
$$

$$
\triangleleft\!\downarrow \frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}
$$

$$
\otimes\!\downarrow \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)}
$$

Figure 2-3: *Systems* $\mathsf{SABVU}^{\downarrow}$ *and* $\mathsf{SABV}^{\downarrow}$

Example 2.17. $\mathsf{SAMLLS}^{\downarrow}$ is splittable, and the minimal relation + introducing dualities is $\parr$.

Example 2.18. The linear down fragment of classical logic $\mathsf{SAKS}^{\downarrow}$ of Figure 2-2 together with the equality rules corresponding to the axioms of example 1.7 is splittable. The minimal relation + introducing dualities is $\vee$.

Example 2.19. The down fragment of SABVU given in Figure 2-3 $\mathsf{SABVU}^{\downarrow}$ together with the equality rules corresponding to the axioms of example 1.9 is splittable. The minimal relation + introducing dualities is $\parr$.

Likewise, the down fragment of SABV given in the same figure is splittable.

Remark 2.20. From condition 3 in Definition 2.16 and the closure of = under negation, $\times$ is associative and commutative.

Notation 2.21. As all relations $\alpha \neq +$ are non-contractive, all the inference rules of a splittable system are of the form

$$
\alpha\!\downarrow \frac{(A + B)\; \alpha\; (C + D)}{(A\; \alpha\; C) + (B\; \alpha^m\; D)} \quad .
$$

We denote this rule by $\alpha\!\downarrow$.

The idea behind the generalisation of splitting is simple: if a relation α is non-contractive, its scope only widens when following it from the bottom to the top of a proof. Therefore, given a proof

$$
\begin{array}{c} \phi \,\| \\ S\{A\; \alpha\; B\} \end{array} \quad ,
$$

we can follow α all the way to the top of π we will find that its scope only widens and that ϕ is of the form

$$\alpha\downarrow \frac{\boxed{\begin{array}{c}\Pi \\ A+Q_1\end{array}} \ \alpha \ \boxed{\begin{array}{c}\Pi \\ B+Q_2\end{array}}}{\boxed{\begin{array}{c}(A\ \alpha\ B)+(Q_1\ \alpha^m\ Q_2) \\ \| \\ S\{A\ \alpha\ B\}\end{array}}} \quad .$$

In other words, the proof ϕ splits into two subproofs that have no interaction above $\alpha\downarrow$.

We will obtain the admissibility of certain rules as a corollary of splitting. In particular, we will show that the subatomic rule that corresponds to the atomic cut rule is admissible. To prove that this result corresponds to cut-elimination in the original systems, we will need to show that the cut-free proofs obtained from proofs of the non-subatomic original system via this procedure are interpretable themselves, and therefore correspond to proofs in the original system. For that, we will pay particular attention to tame proofs, in which no inference rule occurs in the scope of an atom. If the interpretation I is built in a natural way, every proof of the original system will be represented by a tame proof in SA. The interpretability of tame proofs is preserved by splitting as long as interpretability is preserved by duals. In that case, as a corollary, interpretability will be preserved by the cut-elimination procedure.

Definition 2.22. We define $=_+$ as the equivalence relation on formulae defined by the axioms for the associativity, commutativity, unit of $+$ and constant assignments for $+$.

We define the equivalence relation $=_+$ on derivations following Definition 1.32.

Definition 2.23. We say that a system SA with a natural interpretation I, negation $\bar{\cdot}$ and an equational theory $=$ is *preservable* when:

1. If A is interpretable and $A =_+ B$, then B is interpretable ;
2. If $A\ \alpha\ B$ is interpretable, $\alpha \in \mathscr{R}$, then A and B are interpretable ;
3. If $A\ a\ B$ is interpretable and $A + A' = 1$, $B + B' = 1$ then $A'\ a\ B'$ is interpretable for $a \in \mathscr{A}$;
4. If A is interpretable, then $\overline{A}$ is interpretable ;
5. The atoms of $\mathscr{A}$ are non-commutative, non-associative and non-unitary.

These conditions ensure that interpretability is preserved by duality, meaning that if an instance of a rule is interpretable, the same rule instantiated with the duals of the formulae involved is interpretable as well.

The proof of the splitting result is done in two steps for ease of reading: shallow splitting and context reduction, just as in the example in Section 2.1. As noted in [26] and in [40], the main difficulty of splitting is finding the right induction measure for every system. In the literature, each splitting theorem for each proof system uses a different induction measure tailored specifically for it. By providing a general splitting theorem, we not only give a formal definition of what a splitting theorem is, but also give a new one-size-fits-all induction measure that works for every splittable system, taking the search for an induction measure out of the process for designing a proof system.

Lemma 2.24. *If* $\mathsf{SA}^{\downarrow}$ *is splittable, then for every proof*

$$\begin{array}{c} \phi \,\Vert\, \mathsf{SA}^{\downarrow} \\ u + C \end{array}$$

where $u \in \mathcal{U}$*, there is a derivation*

$$\begin{array}{c} \bar{u} \\ \psi \,\Vert\, \mathsf{SA}^{\downarrow} \\ C \end{array} .$$

Furthermore, if $\mathsf{SA}^{\downarrow}$ *is preservable, then if* ϕ *is tame we have that* ψ *is tame.*

Proof. We take

$$\psi \equiv {}^{\times\downarrow}\frac{(\bar{u} + 0) \times \boxed{\begin{array}{c} \phi \,\Vert \\ u + C \end{array}}}{\boxed{=\frac{\bar{u} \times u}{0}} + 0 + C} .$$

□

Definition 2.25. Given a derivation ϕ, we define the *length* of ϕ as the number of rules in ϕ different from the equality rules for the associativity and commutativity of $+$, the unit rule for $+$ and the unit assignments for $+$. We denote it by $|\phi|_+$.

It is straightforward that if $\phi =_+ \psi$, then $|\phi|_+ = |\psi|_+$. It is clear as well that if SA is preservable and ϕ is tame, then ψ is as well, since interpretability is preserved by $=_+$ and we cannot add or remove non-equality rules in the scope of atoms from a formula through the equalities of $=_+$.

Notation 2.26. We will abuse notation and refer to derivations made up only of equality rules rules as *equalities.*

Theorem 2.27 (Shallow Splitting)**.** *If* $\mathsf{SA}^{\downarrow}$ *is splittable, for every formulae* A*,* B*,* C*, for every relation* $\alpha \neq +$*, for every proof*

$$\begin{array}{c} \phi \,\Vert\, \mathsf{SA}^{\downarrow} \\ (A \;\alpha\; B) + C \end{array}$$

there exist formulae Q_1, Q_2 and derivations

$$\begin{array}{c} Q_1 \; \bar{\alpha} \; Q_2 \\ \psi \, \| \, \mathsf{SA}^{\downarrow} \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ A + Q_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi|_+$.

If $SA^{\downarrow}$ is preservable and ϕ is tame, then ϕ_1, ϕ_2 and ψ are tame. Furthermore, if α is an atom then ϕ_1 and ϕ_2 are equalities.

Proof. Given a proof ϕ in SA of $(A \; \alpha \; B) + C$ we reduce it to CoS notation for $=_+$. We will proceed by induction on $|\phi|_+$.

If $|\phi|_+ = 1$, then $A =_+ v, B =_+ w$ and $v \; \alpha \; w =_+ u$, with $u + C =_+ 1$. By Lemma 2.24, there is a derivation $\begin{array}{c} \bar{u} \\ \psi' \, \| \, \mathsf{SA}^{\downarrow} \\ C \end{array}$ and we take:

$$\psi \equiv \begin{array}{c} = \dfrac{\bar{v} \; \bar{\alpha} \; \bar{w}}{\bar{u}} \\ \psi' \, \| \\ C \end{array} \quad , \quad \phi_1 \equiv \; =_+ \frac{1}{\boxed{=_+ \dfrac{v}{A}} + \bar{v}} \quad \text{and} \quad \phi_2 \equiv \; =_+ \frac{1}{\boxed{=_+ \dfrac{w}{B}} + \bar{w}} \quad .$$

ψ' is tame and $\bar{v} \; \bar{\alpha} \; \bar{w}$ is interpretable, and therefore ψ is tame. Furthermore, ϕ_1 and ϕ_2 are tame and equalities.

If $|\phi|_+ = |\phi'|_+ > 1$, we prove the inductive step for all the possible cases of the bottom inference rule ρ of ϕ.

Inspection of the rules provides us with the following possible cases:

$$(1) \;\; \phi =_+ \; \rho \, \frac{\begin{array}{c} \phi' \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ (A \; \alpha \; B) + C' \end{array}}{(A \; \alpha \; B) + C} \quad ;$$

$$(2) \;\; \phi =_+ \; \times\!\!\downarrow \frac{\begin{array}{c} \phi' \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ (((A \; \alpha \; B) + C_1) \times (C_2 + C_3)) + C_4 \end{array}}{(A \; \alpha \; B) + C_2 + (C_1 \times C_3) + C_4} \quad ;$$

$$(3) \;\; \phi =_+ \; = \frac{\begin{array}{c} \phi' \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ (((A \; \alpha \; B) + C_1) \; \beta \; u_\beta) + C_2 \end{array}}{(A \; \alpha \; B) + C_1 + C_2} \quad ;$$

$$(4) \;\; \phi =_+ \; = \frac{\begin{array}{c} \phi' \, \overline{\|} \, \mathsf{SA}^{\downarrow} \\ (u_\beta \; \beta \; ((A \; \alpha \; B) + C_1)) + C_2 \end{array}}{(A \; \alpha \; B) + C_1 + C_2} \quad ;$$

$$(5)\quad \phi =_{+} {}_{\rho}\frac{(A' \;\alpha\; B) + C}{(A \;\alpha\; B) + C}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad ;$$

$$(6)\quad \phi =_{+} {}_{\rho}\frac{(A \;\alpha\; B') + C}{(A \;\alpha\; B) + C}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad ;$$

$$(7)\quad \phi =_{+} {}_{\alpha\downarrow}\frac{((A + C_1) \;\alpha\; (B + C_2)) + C_3}{((A \;\alpha\; B) + (C_1 \;\overline{\alpha}\; C_2)) + C_3}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{if } \alpha \text{ is strong ;}$$

$$(8)\quad \phi =_{+} {}_{\alpha\downarrow}\frac{((A + C_1) \;\overline{\alpha}\; (B + C_2)) + C_3}{((A \;\alpha\; B) + (C_1 \;\overline{\alpha}\; C_2)) + C_3}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{if } \alpha \text{ is weak ;}$$

$$(9)\quad \phi =_{+} {}_{\alpha\downarrow}\frac{((A + C_1) \;\alpha\; (B + C_2)) + C_3}{((A \;\alpha\; B) + (C_1 \;\alpha\; C_2)) + C_3}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{if } \alpha \text{ is weak ;}$$

$$(10)\quad \phi =_{+} {}_{=}\frac{(B \;\alpha\; A) + C}{(A \;\alpha\; B) + C}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{if } \alpha \text{ is commutative ;}$$

$$(11)\quad \phi =_{+} {}_{=}\frac{((A \;\alpha\; B_1) \;\alpha\; B_2) + C}{(A \;\alpha\; (B_1 \;\alpha\; B_2)) + C}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{if } \alpha \text{ is associative ;}$$

$$(12)\quad \phi =_{+} {}_{=}\frac{(A_1 \;\alpha\; (A_2 \;\alpha\; B)) + C}{((A_1 \;\alpha\; A_2) \;\alpha\; B) + C}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{if } \alpha \text{ is associative ;}$$

$$(13)\quad \phi =_{+} {}_{=}\frac{A + C}{(A \;\alpha\; u_\alpha) + C}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{if } \alpha \text{ is unitary, with } B =_{+} u_\beta \text{ ;}$$

$$(14)\quad \phi =_{+} {}_{=}\frac{B + C}{(u_\alpha \;\alpha\; B) + C}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{if } \alpha \text{ is unitary, with } A =_{+} u_\beta \text{ ;}$$

$$(15)\quad \phi =_{+} {}_{=}\frac{u + C}{(v \;\alpha\; w) + C}^{\,\phi' \,\overline{\|}\, \mathsf{SA}^{\downarrow}} \qquad \text{with } A =_{+} v \text{ and } B =_{+} w \text{ .}$$

We proceed as follows:

(1) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$.

There are derivations

$$\psi =_+ \begin{array}{c} Q_1 \; \overline{\alpha} \; Q_2 \\ {\scriptstyle \psi' \, \| \, \mathsf{SA}^{\downarrow}} \\ C' \\ {\scriptstyle \rho}\,\overline{} \\ C \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \phi_1 \, \overline{\|} \, \mathsf{SA}^{\downarrow}} \\ A + Q_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} {\scriptstyle \phi_2 \, \overline{\|} \, \mathsf{SA}^{\downarrow}} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi|_+ < |\phi|_+$.

If ϕ is tame, then ρ and ϕ_1, ϕ_2 and ψ' are tame. Hence ψ is tame.

Furthermore, if α is an atom then by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(2) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$.

There are derivations

$$\psi =_+ \begin{array}{c} H_1 + H_2 \\ {\scriptstyle \psi' \, \| \, \mathsf{SA}^{\downarrow}} \\ C_4 \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \omega_1 \, \overline{\|} \, \mathsf{SA}^{\downarrow}} \\ (A \; \alpha \; B) + C_1 + H_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} {\scriptstyle \omega_2 \, \overline{\|} \, \mathsf{SA}^{\downarrow}} \\ C_2 + C_3 + H_2 \end{array} \quad ,$$

with $|\omega_1|_+ + |\omega_2|_+ \leq |\phi''|_+$.

If ϕ is tame, then ϕ' is tame and ω_1, ω_2 and ψ' are tame.

We apply the induction hypothesis to ω_1 as $|\omega_1|_+ \leq |\phi'|_+ < |\phi|_+$.

There are derivations

$$\begin{array}{c} Q_1 \; \overline{\alpha} \; Q_2 \\ {\scriptstyle \psi'' \, \| \, \mathsf{SA}^{\downarrow}} \\ C_1 + H_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1 \, \overline{\|} \, \mathsf{SA}^{\downarrow}} \\ A + Q_1 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_2 \, \overline{\|} \, \mathsf{SA}^{\downarrow}} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\omega_1|_+ < |\phi|_+$.

We take:

$$\psi =_+ \times\!\!\downarrow \frac{\boxed{\begin{array}{c} Q_1 \; \overline{\alpha} \; Q_2 \\ {\scriptstyle \psi'' \, \|} \\ C_1 + H_1 \end{array}} \times \boxed{\begin{array}{c} {\scriptstyle \omega_2 \, \overline{\|}} \\ C_2 + C_3 + H_2 \end{array}}}{(C_1 \times C_3) + C_2 + \boxed{\begin{array}{c} H_1 + H_2 \\ {\scriptstyle \psi' \, \|} \\ C_4 \end{array}}} \quad .$$

If ϕ is tame, then ω_1 is tame and ϕ_1, ϕ_2 and ψ'' are tame. ψ' and ω_2 are tame as well, and since I is preservable, C_1, C_2, C_3 are interpretable. Therefore ψ is tame.

Furthermore, if α is an atom then by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(3) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1\ \overline{\beta}\ H_2 \\ \psi' \,\|\, \mathsf{SA}^{\downarrow} \\ C_2 \end{array} \quad , \quad \begin{array}{c} \omega_1 \,\overline{\|}\, \mathsf{SA}^{\downarrow} \\ (A\ \alpha\ B) + C_1 + H_1 \end{array} \quad , \quad \begin{array}{c} \omega_2 \,\overline{\|}\, \mathsf{SA}^{\downarrow} \\ u_\beta + H_2 \end{array} \quad ,$$

with $|\omega_1|_+ + |\omega_2|_+ \leq |\phi'|_+$.

By Lemma 2.24, there is a derivation

$$\begin{array}{c} \bar{u}_\beta \\ \psi'' \,\|\, \mathsf{SA}^{\downarrow} \\ H_2 \end{array} \quad .$$

If ϕ is tame, then ω_2 is tame and thus ψ'' is tame.

We apply the induction hypothesis to ω_1 as $|\omega_1|_+ \leq |\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} Q_1\ \overline{\alpha}\ Q_2 \\ \psi''' \,\|\, \mathsf{SA}^{\downarrow} \\ C_1 + H_1 \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\overline{\|}\, \mathsf{SA}^{\downarrow} \\ A + Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_2 \,\overline{\|}\, \mathsf{SA}^{\downarrow} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\omega_1|_+ < |\phi|_+$.

We take:

$$\psi =_+ \begin{array}{c} Q_1\ \overline{\alpha}\ Q_2 \\ \psi''' \,\| \\ C_1 + \boxed{\begin{array}{c} H_1\ \overline{\beta}\ \boxed{\begin{array}{c} \bar{u}_\beta \\ \psi'' \,\| \\ H_2 \end{array}} \\ \psi' \,\| \\ C_2 \end{array}} \end{array} \quad .$$

Atoms are not unitary, and thus β is not an atom. If ϕ is tame, then ω_1 is tame and ϕ_1, ϕ_2 and ψ''' are tame. ψ'' and ψ' are tame as well, and hence ψ is tame.

Furthermore, if α is an atom then by the induction hypothesis ϕ_1 and ϕ_2 are equalities.

(4) This case is analogous to (3).

(5) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} Q_1 \; \alpha^m \; Q_2 \\ {\scriptstyle \psi \,\|\, \mathsf{SA}^{\downarrow}} \\ C \end{array} \quad , \qquad \phi_1 \equiv \begin{array}{c} {\scriptstyle \phi_1' \,\widetilde{\|}\, \mathsf{SA}^{\downarrow}} \\ \boxed{\rho \, \dfrac{A'}{A}} + Q_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} {\scriptstyle \phi_2 \,\widetilde{\|}\, \mathsf{SA}^{\downarrow}} \\ B + Q_2 \end{array} \quad ,$$

with $|\phi_1'|_+ + |\phi_2|_+ \leq |\phi'|_+$.

We have $|\phi_1|_+ + |\phi_2|_+ = |\phi_1'|_+ + 1 + |\phi_2|_+ \leq |\phi'|_+ + 1 = |\phi|_+$.

If ϕ is tame, then ϕ' is tame and ϕ_1', ϕ_2 and ψ are tame. ρ is tame as well, and thus ϕ_1 is tame.

Furthermore, if α is an atom the only allowed instances of ρ are equalities and ϕ_1' is an equality, and thus ϕ_1 is an equality. By induction hypothesis, ϕ_2 is an equality.

(6) This case is analogous to (5).

(7) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \; \overline{\alpha} \; H_2 \\ {\scriptstyle \psi' \,\|\, \mathsf{SA}^{\downarrow}} \\ C_3 \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \phi_1 \,\widetilde{\|}\, \mathsf{SA}^{\downarrow}} \\ A + C_1 + H_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} {\scriptstyle \phi_2 \,\widetilde{\|}\, \mathsf{SA}^{\downarrow}} \\ B + C_2 + H_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi'|_+ < |\phi|_+$.

We take $Q_1 \equiv C_1 + H_1$, $Q_2 \equiv C_2 + H_2$ and

$$\psi =_+ \; \overline{\alpha}{\downarrow} \, \dfrac{(C_1 + H_1) \; \overline{\alpha} \; (C_2 + H_2)}{(C_1 \; \overline{\alpha} \; C_2) + \boxed{\begin{array}{c} H_1 \; \overline{\alpha} \; H_2 \\ {\scriptstyle \psi' \,\|} \\ C_3 \end{array}}} \; .$$

If ϕ is tame, then ϕ' is tame and by induction hypothesis ϕ_1, ϕ_2 and ψ' are tame.

If α is an atom, then by the induction hypothesis ϕ_1 and ϕ_2 are equalities. Then $(C_1 + H_1) \; \overline{\alpha} \; (C_2 + H_2)$ is interpretable by condition 3 of preservability. Therefore, ψ is tame.

If ϕ is tame and α is not an atom, then ψ is trivially tame since C_1, H_1, C_2, H_2 are interpretable and ψ' is tame.

(8) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \;\alpha\; H_2 \\ {\scriptstyle \psi'} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ C_3 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \overline{\Vert} {\scriptstyle \mathsf{SA}^{\downarrow}} \\ A + C_1 + H_1 \end{array} \quad \text{and} \quad \begin{array}{c} {\scriptstyle \phi_2} \overline{\Vert} {\scriptstyle \mathsf{SA}^{\downarrow}} \\ B + C_2 + H_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi'|_+ < |\phi|_+$.

We take $Q_1 \equiv C_1 + H_1$, $Q_2 \equiv C_2 + H_2$ and

$$\psi =_+ \quad \overline{\alpha}\!\downarrow \frac{(C_1 + H_1) \;\overline{\alpha}\; (C_2 + H_2)}{(C_1 \;\overline{\alpha}\; C_2) + \boxed{\begin{array}{c} H_1 \;\alpha\; H_2 \\ {\scriptstyle \psi'} \Vert \\ F_3 \end{array}}} \quad .$$

If ϕ is tame, then ϕ' is tame and by induction hypothesis ϕ_1, ϕ_2 and ψ' are tame.

If α is an atom, then by the induction hypothesis ϕ_1 and ϕ_2 are equalities. Then $(C_1 + H_1) \;\overline{\alpha}\; (C_2 + H_2)$ is interpretable by condition 3 of preservability. Therefore, ψ is tame.

If ϕ is tame and α is not an atom, then ψ is trivially tame since C_1, H_1, C_2, H_2 are interpretable and ψ' is tame.

(9) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \;\overline{\alpha}\; H_2 \\ {\scriptstyle \psi'} \Vert {\scriptstyle \mathsf{SA}^{\downarrow}} \\ F_3 \end{array} \quad , \quad \begin{array}{c} {\scriptstyle \phi_1} \overline{\Vert} {\scriptstyle \mathsf{SA}^{\downarrow}} \\ A + C_1 + H_1 \end{array} \quad \text{and} \quad \begin{array}{c} {\scriptstyle \phi_2} \overline{\Vert} {\scriptstyle \mathsf{SA}^{\downarrow}} \\ B + C_2 + H_2 \end{array} \quad ,$$

with $|\phi_1|_+ + |\phi_2|_+ \leq |\phi'|_+ < |\phi|_+$.

We take $Q_1 \equiv C_1 + H_1$, $Q_2 \equiv C_2 + H_2$ and

$$\psi =_+ \quad \alpha^m\!\downarrow \frac{(C_1 + H_1) \;\overline{\alpha}\; (C_2 + H_2)}{(C_1 \;\alpha\; C_2) + \boxed{\begin{array}{c} H_1 \;\overline{\alpha}\; H_2 \\ {\scriptstyle \psi'} \Vert \\ C_3 \end{array}}} \quad .$$

If ϕ is tame, then ϕ' is tame and by induction hypothesis ϕ_1, ϕ_2 and ψ' are tame.

If α is an atom, then by the induction hypothesis ϕ_1 and ϕ_2 are equalities. Then $(C_1 + H_1) \;\overline{\alpha}\; (C_2 + H_2)$ is interpretable by condition 3 of preservability. Therefore, ψ is tame.

If ϕ is tame and α is not an atom, then ψ is trivially tame since C_1, H_1, C_2, H_2 are interpretable and ψ' is tame.

(10) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \mathbin{\overline{\alpha}} H_2 \\ {\scriptstyle \psi'} \, \| \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ C \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \omega_1} \, \overline{\|} \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ B + H_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} {\scriptstyle \omega_2} \, \overline{\|} \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ A + H_2 \end{array} \quad ,$$

with $|\omega_1|_+ + |\omega_2|_+ \leq |\phi'|_+$.

We take $Q_1 \equiv H_2$, $Q_2 \equiv H_1$, $\phi_1 \equiv \omega_2$, $\phi_2 \equiv \omega_1$ and

$$\psi \equiv \begin{array}{c} \boxed{= \dfrac{H_2 \mathbin{\overline{\alpha}} H_1}{H_1 \mathbin{\overline{\alpha}} H_2}} \\ {\scriptstyle \psi'} \, \| \\ C \end{array} \quad .$$

Atoms are not commutative and thus α is not an atom.

If ϕ is tame, then ϕ' is tame and by induction hypothesis ψ_1, ψ_2 and ψ' are tame. Then H_1 and H_2 are interpretable and hence ψ is tame as well.

(11) We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} H_1 \mathbin{\overline{\alpha}} H_2 \\ {\scriptstyle \psi'} \, \| \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ C \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \omega_1} \, \overline{\|} \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ (A \mathbin{\alpha} B_1) + H_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} {\scriptstyle \omega_2} \, \overline{\|} \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ B_2 + H_2 \end{array} \quad ,$$

with $|\omega_1|_+ + |\omega_2|_+ \leq |\phi'|_+$.

If ϕ is tame, then ϕ' is tame and by induction hypothesis ω_1, ω_2 and ψ' are tame.

We apply the induction hypothesis to ω_1 as $|\omega_1|_+ \leq |\phi'|_+ < |\phi|_+$. There are

$$\begin{array}{c} Q_1 \mathbin{\overline{\alpha}} H_3 \\ {\scriptstyle \psi''} \, \| \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ H_1 \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \phi_1} \, \overline{\|} \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ A + Q_1 \end{array} \quad , \qquad \begin{array}{c} {\scriptstyle \omega_3} \, \overline{\|} \, {\scriptstyle \mathsf{SA}^{\downarrow}} \\ B_1 + H_3 \end{array} \quad ,$$

with $|\phi_1|_+ + |\omega_3|_+ \leq |\omega_1|_+$.

We take $Q_2 \equiv H_3 \mathbin{\overline{\alpha}} H_2$ and

$$
\phi_2 \equiv {\scriptstyle \alpha\downarrow}\frac{\boxed{\begin{array}{c}{\scriptstyle \omega_3}\,\overline{\Vert} \\ B_1 + H_3\end{array}}\ \alpha^M\ \boxed{\begin{array}{c}{\scriptstyle \omega_2}\,\overline{\Vert} \\ B_2 + H_2\end{array}}}{(B_1 \ \alpha\ B_2) + (H_3\ \overline{\alpha}\ H_2)} \quad , \quad \psi \equiv \begin{array}{c} \boxed{{\scriptstyle =}\frac{Q_1\ \overline{\alpha}\ (H_3\ \overline{\alpha}\ H_2)}{\boxed{\begin{array}{c} Q_1\ \overline{\alpha}\ H_3 \\ {\scriptstyle \psi''}\,\Vert \\ H_1\end{array}}\ \overline{\alpha}\ H_2}} \\ {\scriptstyle \psi'}\,\Vert \\ C \end{array} \quad .
$$

We have $|\phi_1|_+ + |\phi_2|_+ = |\phi_1|_+ + |\omega_3|_+ + |\omega_2|_+ + 1 \leq |\omega_1|_+ + |\omega_2|_+ + 1 \leq |\phi'|_+ + 1 = |\phi|_+$.

Atoms are not associative, thus α is not an atom. If ϕ is tame, then ω_2, ω_3, ψ' and ψ'' are tame and so Q_1, H_2, H_3 are interpretable. Therefore ϕ_1, ϕ_2 and ψ are tame.

(12) This case is analogous to (11).

(13) We take $Q_1 \equiv C$, $Q_2 \equiv \bar{u}_\alpha$ and

$$
\psi \equiv {\scriptstyle =}\frac{C\ \overline{\alpha}\ \bar{u}_\alpha}{C} \quad , \quad \phi_1 \equiv \begin{array}{c}{\scriptstyle \phi'}\,\overline{\Vert} \\ A + C\end{array} \quad , \phi_2 \equiv {\scriptstyle =_+}\frac{1}{\boxed{{\scriptstyle =_+}\frac{u_\alpha}{B}} + \bar{u}_\alpha} \quad .
$$

Then, $|\phi_1|_+ + |\phi_2|_+ = |\phi'|_+ < |\phi|_+$.

If ϕ is tame, then C is interpretable and ϕ' is tame, and therefore ϕ_1, ϕ_2 and ψ are tame.

(14) This case is analogous to (13).

(15) By Lemma 2.24, there is a derivation $\begin{array}{c} \bar{u} \\ {\scriptstyle \psi'}\,\Vert{\scriptstyle \mathsf{SA}\downarrow} \\ C \end{array}$ and we take:

$$
\psi \equiv \begin{array}{c} {\scriptstyle =}\frac{\bar{v}\ \overline{\alpha}\ \bar{w}}{\bar{u}} \\ {\scriptstyle \psi'}\,\Vert \\ C \end{array} \quad , \quad \phi_1 \equiv {\scriptstyle =_+}\frac{1}{\boxed{{\scriptstyle =_+}\frac{v}{A}} + \bar{v}} \quad \text{and} \quad \phi_2 \equiv {\scriptstyle =_+}\frac{1}{\boxed{{\scriptstyle =_+}\frac{w}{B}} + \bar{w}} \quad .
$$

If ϕ is tame, then ψ' is tame and ϕ_1 and ϕ_2 are tame. Since $v\ \alpha\ w$ is interpretable, by condition 4 of preservability $\bar{v}\ \overline{\alpha}\ \bar{w}$ is interpretable. Therefore ψ is tame. Furthermore, ϕ_1 and ϕ_2 are equalities.

$\square$

We can see that shallow splitting hinges precisely on the non-contractiveness of relations and on the duality between constants.

Remark 2.28. The requirement for $+$ to be associative and commutative can be relaxed, with the condition that the rule $\times\!\downarrow$ be restricted in such a way that it corresponds to two rules

$$\frac{(A+B)\times C}{(A\times C)+B} \quad \text{and} \quad \frac{A\times(B+C)}{B+(A\times C)} \quad .$$

Since all relations are non-contractive, we can apply shallow splitting to the outermost relation in any context S, and continue applying it inductively to split any proof completely. This process is formalised in the following Theorem 2.29, which is a generalisation of Theorem 4.1.5 in [26].

Theorem 2.29 (Context Reduction)**.** *Let* $\mathsf{SA}^{\downarrow}$ *be a splittable system. For any formula* A *and for any context* $S\{\ \}$*, given a proof* $\begin{array}{c}\phi\,\|\,\mathsf{SA}^{\downarrow}\\ S\{A\}\end{array}$*, there exist a formula* K*, a provable context* $H\{\ \}$ *and derivations*

$$\begin{array}{c}\zeta\,\|\,\mathsf{SA}^{\downarrow}\\ A+K\end{array} \quad \textit{and} \quad \begin{array}{c}H\{\{\ \}+K\}\\ \chi\,\|\,\mathsf{SA}^{\downarrow}\\ S\{\ \}\end{array}$$

such that if ϕ *is tame, then* ζ *is tame.*

Furthermore, if $\{\ \}$ *is not in the scope of an atom in* $S\{\ \}$ *and* ϕ *is tame, then* χ *is tame.*

Proof. We proceed by induction on the number of relations $\alpha \neq +$ that $\{\ \}$ is in the scope of in $S\{\ \}$. We denote it by $|S|_+$.

If $|S|_+ = 0$, then $S\{A\} =_+ A+K$ and we take $\zeta =_+ \phi$ and $H\{\ \} = \{\ \}$.

If $S\{A\} =_+ (S'\{A\}\ \beta\ B)+C$ with $\beta \neq +$, we apply Theorem 2.27 to ϕ. There exist derivations

$$\begin{array}{c}Q_1\ \overline{\beta}\ Q_2\\ \psi\,\|\,\mathsf{SA}^{\downarrow}\\ C\end{array} \quad , \quad \begin{array}{c}\phi_1\,\|\,\mathsf{SA}^{\downarrow}\\ S'\{A\}+Q_1\end{array} \quad \text{and} \quad \begin{array}{c}\phi_2\,\|\,\mathsf{SA}^{\downarrow}\\ B+Q_2\end{array}$$

such that ϕ_1, ϕ_2 and ψ are tame if ϕ is tame.

We apply the induction hypothesis to ϕ_1 since $|S'|_+ < |S|_+$. There are derivations

$$\begin{array}{c}\zeta\,\|\,\mathsf{SA}^{\downarrow}\\ A+K\end{array} \quad , \quad \begin{array}{c}H'\{\{\ \}+K\}\\ \chi'\,\|\,\mathsf{SA}^{\downarrow}\\ S'\{\ \}+Q_1\end{array} \quad ,$$

with H' a provable context, such that ζ is tame if ϕ_1 is tame.

We take $H\{\ \} = H'\{\ \}\ \beta^M\ 1$. We have $H\{1\} = H'\{1\}\ \beta^M\ 1 = 1\ \beta^M\ 1 = 1$, and we can build in $\mathsf{SA}^{\downarrow}$

$$\chi \equiv \beta\!\downarrow \frac{\boxed{\begin{array}{c} H'\{\{\ \} + K\} \\ \chi' \| \\ S'\{\ \} + Q_1 \end{array}}\ \beta^M\ \boxed{\begin{array}{c} \phi_2 \| \\ B + Q_2 \end{array}}}{(S'\{\ \}\ \beta\ B) + \boxed{\begin{array}{c} Q_1\ \overline{\beta}\ Q_2 \\ \psi \| \\ C \end{array}}} \quad .$$

If $\{\ \}$ is not in the scope of an atom in $S\{\ \}$ and ϕ is tame, then by the induction hypothesis χ' is tame and $\{\ \}$ is not in the scope of an atom in $H'\{\ \}$. Since β is not an atom, $\{\ \}$ is not in the scope of an atom in $H\{\ \}$ and χ is tame.

We proceed likewise if $S\{A\} =_+ (B\ \beta\ S'\{A\}) + C$.

□

As a corollary of shallow splitting and context reduction we can show the admissibility of a class of up-rules. The main idea is that through splitting we can separate a proof into "building blocks" that are independently provable. We can then easily combine these building blocks differently to obtain a new proof with the same conclusion.

Since tameness is preserved by splitting, cut-free proofs obtained from tame proofs will be tame themselves. The cut-free proofs obtained from non-subatomic proofs will therefore be interpretable, and we can ensure that this cut-elimination result corresponds to cut-elimination in the original system.

When designing a proof system that enjoys cut-elimination, we will therefore only have to ensure that the interpretation map is preservable. This is quite an easy task, since the conditions for an interpretation map to be natural are very lenient, and therefore there is much freedom to design an interpretation to suit many needs.

Definition 2.30. Rules of the form $\alpha\!\uparrow \dfrac{(A\ \alpha\ B) \times \left(C\ \alpha^M\ D\right)}{(A \times C)\ \alpha\ (B \times D)}$ are *cuts*.

Corollary 2.31 (Admissibility of cuts)**.** *Let* SA *be a splittable proof system.*

For any formulae A, B, C, D, any context S, any relation $\alpha \neq +$, given a proof

$$\phi \equiv \begin{array}{c} \phi' \| \mathsf{SA}^{\downarrow} \\ S\left\{ \alpha\!\uparrow \dfrac{(A\ \alpha\ B) \times \left(C\ \alpha^M\ D\right)}{(A \times C)\ \alpha\ (B \times D)} \right\} \end{array} \quad ,$$

there is a proof

$$\begin{array}{c} \pi \| \mathsf{SA}^{\downarrow} \\ S\{(A \times C)\ \alpha\ (B \times D)\} \end{array} \quad .$$

Furthermore, if ϕ is tame and α is not an atom, π is tame.

Proof. We apply Theorem 2.29 to ϕ.

There are derivations

$$\begin{array}{c} \zeta \Vert \mathsf{SA}^{\downarrow} \\ \big((A \; \alpha \; B) \times \big(C \; \alpha^M \; D\big)\big) + K \end{array} \qquad \text{and} \qquad \begin{array}{c} H\{\{\ \} + K\} \\ \chi \Vert \mathsf{SA}^{\downarrow} \\ S\{\ \} \end{array} \quad ,$$

with $H\{1\} = 1$.

We apply Theorem 2.27 to ζ. There exist derivations

$$\begin{array}{c} Q_1 + Q_2 \\ \psi \Vert \mathsf{SA}^{\downarrow} \\ K \end{array} \quad , \quad \begin{array}{c} \phi_1 \Vert \mathsf{SA}^{\downarrow} \\ (A \; \alpha \; B) + Q_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \Vert \mathsf{SA}^{\downarrow} \\ \big(C \; \alpha^M \; D\big) + Q_2 \end{array} \quad .$$

We apply Theorem 2.27 to ϕ_3 and ϕ_4 and we obtain

$$\begin{array}{c} Q_A \; \overline{\alpha} \; Q_B \\ \psi_1 \Vert \mathsf{SA}^{\downarrow} \\ Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_3 \Vert \mathsf{SA}^{\downarrow} \\ Q_A + A \end{array} \quad \text{and} \quad \begin{array}{c} \phi_4 \Vert \mathsf{SA}^{\downarrow} \\ Q_B + B \end{array} \quad ,$$

$$\begin{array}{c} Q_C \; \alpha^m \; Q_D \\ \psi_2 \Vert \mathsf{SA}^{\downarrow} \\ Q_2 \end{array} \quad , \quad \begin{array}{c} \phi_5 \Vert \mathsf{SA}^{\downarrow} \\ Q_C + C \end{array} \quad \text{and} \quad \begin{array}{c} \phi_6 \Vert \mathsf{SA}^{\downarrow} \\ Q_D + D \end{array} \quad .$$

We can then build the following proof in $\mathsf{SA}^{\downarrow}$

$$\pi = \begin{array}{c} H\left\{ {\scriptstyle \alpha^M\downarrow} \dfrac{ \boxed{ {\scriptstyle \times\downarrow} \dfrac{ \boxed{\begin{array}{c} \phi_3 \Vert \\ A + Q_A \end{array}} \times \boxed{\begin{array}{c} \phi_5 \Vert \\ C + Q_C \end{array}} }{ (A \times C) + Q_A + Q_C } } \; \alpha^M \; \boxed{ {\scriptstyle \times\downarrow} \dfrac{ \boxed{\begin{array}{c} \phi_4 \Vert \\ B + Q_B \end{array}} \times \boxed{\begin{array}{c} \phi_6 \Vert \\ D + Q_D \end{array}} }{ (B \times D) + Q_B + Q_D } } }{ ((A \times C) \; \alpha \; (B \times D)) + \boxed{\begin{array}{c} {\scriptstyle \overline{\alpha}\downarrow} \dfrac{ (Q_A + Q_C) \; \overline{\alpha} \; (Q_B + Q_D) }{ \boxed{\begin{array}{c} Q_A \; \overline{\alpha} \; Q_B \\ \psi_1 \Vert \\ Q_1 \end{array}} + \boxed{\begin{array}{c} Q_C \; \alpha^m \; Q_D \\ \psi_2 \Vert \\ Q_2 \end{array}} } \\ \psi \Vert \\ K \end{array}} } \right\} \\ \chi \Vert \\ S\{(A \times C) \; \alpha \; (B \times D)\} \end{array} \quad .$$

If ϕ is tame, then $\{\ \}$ is not in the scope of an atom in $S\{\ \}$ and $\phi_3, \phi_4, \phi_5, \phi_6, \psi_1, \psi_2$ and χ are tame. Therefore, if α is not an atom, π is tame.

$\square$

Remark 2.32. The rule $+\!\uparrow \dfrac{(A + B) \times (C \times D)}{(A \times C) + (B \times D)}$ is always admissible in systems with the rule $\times\!\downarrow$ where $\times$ is associative. We obtain it as follows:

$$
=\frac{=\frac{\times\downarrow\frac{=\frac{\times\downarrow\frac{(A+B)\times(C\times D)}{((A+B)\times C)\times D}}{((A+B)\times(C+0))\times D}}{((A\times C)+(B+0))\times D}}{((A\times C)+B)\times(0+D)}}{(A\times C)+0+(B\times D)}\quad .
$$

Example 2.33. We can apply this theorem to show the admissibility of the up fragment of SAMLLS.

Example 2.34. We have shown the admissibility of the up rules

$$
a\uparrow\frac{(A\;a\;B)\wedge(C\;a\;D)}{(A\wedge C)\;a\;(B\wedge D)}\quad\text{and}\quad \vee\uparrow\frac{(A\vee B)\wedge(C\wedge D)}{(A\vee C)\wedge(B\vee D)}
$$

in system $\mathsf{SAKS}^{\downarrow}$.

We can show the admissibility of these rules in system $\mathsf{SAKS}^{\downarrow}$ where $\wedge$ is associative and commutative, or we could use the splitting procedure to show the admissibilty of commutativity and associativity of $\wedge$ as well, if we consider them as given by the rule

$$
\wedge\uparrow\frac{(A\wedge B)\wedge(C\wedge D)}{(A\wedge C)\wedge(B\wedge D)}\quad .
$$

Every rule of the linear fragment of system KS for classical logic corresponds to a tame derivation in SAKS. Therefore every proof in that fragment corresponds to a tame proof in SAKS.

Tameness is preserved when eliminating rule $a\uparrow$ since every instance of a rule $\wedge\downarrow$ with the premiss equal to t has conclusion equal to t and can therefore be replaced by an equality to obtain a tame cut-free proof. Therefore, if α is an atom and ϕ is tame in Theorem 2.31, π is tame as well.

Example 2.35. We have shown the admissibility of the up rules of system SABVU, $a\uparrow$ and $\lhd\uparrow$. Just as above, we can likewise choose to show the admissibility of commutativity and associativity of $\otimes$. The cut-free proofs obtained from tame proofs are tame, since identically to the case of SAMLLS, if there is an interpretable instance of $a\uparrow$, then the instances of $\otimes\downarrow$ in the cut-free proof can be replaced by equalities to obtain a tame proof (see the proof of Theorem 2.11).

This extends to system BV where the units are identified. Even though system SABV does not verify condition 3 of preservability, in a tame proof there are no instances of the equality axioms $0=\circ$ and $1=\circ$ in the scope of an atom since $\circ$ in the scope of an atom is not interpretable. Therefore, in Theorem 2.27, if ϕ is tame and α is an atom then ϕ_1 and ϕ_2 are equalities that do not contain any instance of these axioms. Tameness is preserved since in the absence of these axioms condition 3 of preservability holds.

The splitting procedure is therefore a very general phenomenon: it can be applied to systems with any number of relations and units as long as certain basic equations are satisfied, and is maintained by the identification of any of these units.

2.3 The robustness of splitting: adding a modality

As we have shown in the previous section, splitting hinges only on the shape of rules and on dualities. In the general splitting theorem that we presented we considered only binary relations, but it will be the focus of future research to extend this result to include relations of different arities: splitting can be applied to different types of unary operators, as is shown by the splitting theorems for exponentials in [40] or for a self-dual binder in [38]. In this section we will show a starting point in the direction of such a generalisation, by extending the general procedure to a system with a self-dual modality. The fact that it is possible to do so shows the robustness of the general splitting methodology: it is based on properties that are present in systems with very different expressiveness and therefore it can be expanded to include an extremely wide variety of relations as long as they are introduced by rules of non-contractive shape.

We will present system SAKV^- [27], a system with a self-dual modality. SAKV^- combines a linear splittable core with a self-dual commutative connective (therefore being outside the realm of what is achievable with Gentzen-style calculi) and the simplest case of a modality in terms of the further study of decomposition, the self-dual modality $\star$.

Definition 2.36. We define the set $\mathscr{R} = \mathscr{A} \cup \{⅋, \triangleleft, \otimes\}$ where $\mathscr{A}$ is a denumerable set with $\mathscr{A} \cap \{⅋, \triangleleft, \otimes\} = \varnothing$. We define the set $\mathscr{U} = \{\bot, \circ, 1\}$ of constants. The set $\mathscr{F}$ of *formulae* of SAKV^- contains terms defined by the grammar

$$\mathscr{F} ::= \mathscr{U} \mid \star\mathscr{F} \mid \mathscr{F}\ \alpha\ \mathscr{F} \quad ,$$

with $\alpha \in \mathscr{R}$.

We define *negation* as an involutive map $\bar{\cdot}$ on $\mathscr{F}$ by setting:

$$\begin{aligned}
&\bar{\otimes} := ⅋ \ ; \\
&\bar{\triangleleft} := \triangleleft \ ; \\
&\bar{a} := a \text{ for all } a \in \mathscr{A} \ ; \\
&\bar{\circ} := \circ \ ; \\
&\bar{\bot} := 1
\end{aligned}$$

$$
\begin{array}{cc}
a{\downarrow}\,\dfrac{(A \parr B)\; a\; (C \parr D)}{(A\; a\; C) \parr (B\; a\; D)} & a{\uparrow}\,\dfrac{(A\; a\; B) \otimes (C\; a\; D)}{(A \otimes C)\; a\; (B \otimes D)} \\
\otimes{\downarrow}\,\dfrac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} & \parr{\uparrow}\,\dfrac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)} \\
\lhd{\downarrow}\,\dfrac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} & \lhd{\uparrow}\,\dfrac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)} \\
\star{\downarrow}\,\dfrac{\star(A \parr B)}{\star A \parr \star B} & \star{\uparrow}\,\dfrac{\star A \otimes \star B}{\star(A \otimes B)}
\end{array}
$$

Figure 2-4: *System* SAKV^-

and

$$
\begin{aligned}
&\overline{A\ \alpha\ B} := \overline{A}\ \overline{\alpha}\ \overline{B}\ ; \\
&\overline{\star A} := \star \overline{A}\ .
\end{aligned}
$$

We define an equational theory $=$ on $\mathscr{F}$ as the minimal equivalence relation closed under negation and under context defined by:

For all $A, B, C \in \mathscr{F}$:

$$
\begin{array}{ll}
(A \otimes B) \otimes C = A \otimes (B \otimes C)\ ; & (A \parr B) \parr C = A \parr (B \parr C)\ ; \\
A \otimes B = B \otimes A\ ; & A \parr B = B \parr A\ ; \\
(A \lhd B) \lhd C = A \lhd (B \lhd C)\ ; & \\
A \otimes 1 = A\ ; & A \parr \bot = A\ ; \\
A \lhd \circ = A\ ; & \circ \lhd A = A\ ; \\
\circ \otimes \circ = \bot\ ; & \circ \parr \circ = 1\ ; \\
\bot \lhd \bot = \bot\ ; & 1 \lhd 1 = 1\ ; \\
\forall a \in \mathscr{A}.\ \bot\ a\ \bot = \bot\ ; & \forall a \in \mathscr{A}.\ 1\ a\ 1 = 1\ ; \\
\star\circ = \circ\ ; & \\
1 = \circ\ ; & \bot = \circ\ .
\end{array}
$$

The subatomic proof system SAKV^- is given by the inference rules in Figure 2-4, together with the equality rules given by $=\dfrac{A}{B}$ for every A, B on opposite sides of the equality axioms above.

A proof in SAKV^- is a derivation with premiss 1.

We define $\mathsf{SAKV}^{\downarrow}$ as the system given by the down-rules of system SAKV^-.

We can observe that the rules $\star{\downarrow}$ and $\star{\uparrow}$ correspond to the unary versions of the rules $\alpha{\downarrow}$ considered in the previous section. Furthermore, the constants verify the same equations than for BV and therefore they verify the duality conditions necessary for the splitting theorem. For these reasons, extending this result to SKV^- is a straightforward task, showcasing the generality of the conditions that allow us to obtain splitting.

For the sake of brevity we omit considerations about tameness, that are done identically to the previous section.

Theorem 2.37.

1. *For every formulae A, B, C, for every relation $\alpha \neq \parr$, for every proof*

$$\begin{array}{c} \phi \,\|\, \mathsf{SAKV}^{\downarrow} \\ (A \;\alpha\; B) \parr C \end{array}$$

there exist formulae Q_1, Q_2 and derivations

$$\begin{array}{c} Q_1 \;\overline{\alpha}\; Q_2 \\ \psi \,\|\, \mathsf{SAKV}^{\downarrow} \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\|\, \mathsf{SAKV}^{\downarrow} \\ A \parr Q_1 \end{array} \quad \text{and} \quad \begin{array}{c} \phi_2 \,\|\, \mathsf{SAKV}^{\downarrow} \\ B \parr Q_2 \end{array} \quad ,$$

with $|\phi_1|_{\parr} + |\phi_2|_{\parr} \leq |\phi|_{\parr}$.

2. *For every formulae A, C, for every proof*

$$\begin{array}{c} \phi \,\|\, \mathsf{SAKV}^{\downarrow} \\ \star A \parr C \end{array}$$

there exists a formula Q and derivations

$$\begin{array}{c} \star Q \\ \psi \,\|\, \mathsf{SAKV}^{\downarrow} \\ C \end{array} \quad , \quad \begin{array}{c} \phi_1 \,\|\, \mathsf{SAKV}^{\downarrow} \\ A \parr Q \end{array} \quad ,$$

with $|\phi_1|_{\parr} \leq |\phi|_{\parr}$.

Proof.

1. This case is an instance of the general splitting theorem 2.27, since it is straightforward that the presence of rule $\star\!\downarrow$ does not introduce any new cases and that the conditions are satisfied.

2. We proceed by induction on $|\phi|_{\parr}$. The base case is an instance of case (7) below.

   We prove the inductive step for all the possible cases of the bottom inference rule ρ of ϕ.

   Identically to the proof of 2.27, inspection of the rules provides us with the following possible cases:

(1) $\phi =_{\parr} \; \rho \, \dfrac{\overset{\phi' \,\|\, \mathsf{SAKV}^{\downarrow}}{\star A \parr C'}}{\star A \parr C}$;

(2) $\phi =_{\parr} \; {\otimes\!\downarrow} \, \dfrac{\overset{\phi' \,\|\, \mathsf{SAKV}^{\downarrow}}{((\star A \parr C_1) \otimes (C_2 \parr C_3)) \parr C_4}}{\star A \parr C_2 \parr (C_1 \otimes C_3) \parr C_4}$;

(3) $\phi =_{\parr} \; = \, \dfrac{\overset{\phi' \,\|\, \mathsf{SAKV}^{\downarrow}}{((\star A \parr C_1) \; \beta \; u_\beta) \parr C_2}}{\star A \parr C_1 \parr C_2}$;

(4) $\phi =_{\parr} \; = \, \dfrac{\overset{\phi' \,\|\, \mathsf{SAKV}^{\downarrow}}{(u_\beta \; \beta \; (\star A \parr C_1)) \parr C_2}}{\star A \parr C_1 \parr C_2}$;

(5) $\phi =_{\parr} \; \rho \, \dfrac{\overset{\phi' \,\|\, \mathsf{SAKV}^{\downarrow}}{\star A' \parr C}}{\star A \parr C}$;

(6) $\phi =_{\parr} \; {\parr\!\downarrow} \, \dfrac{\overset{\phi' \,\|\, \mathsf{SAKV}^{\downarrow}}{\star (A \parr C_1) \parr C_2}}{\star A \parr \star C_1 \parr C_2}$;

(7) $\phi =_{\parr} \; = \, \dfrac{\overset{\phi' \,\|\, \mathsf{SAKV}^{\downarrow}}{\circ \parr C}}{\star \circ \parr C}$;

We proceed as follows:

(1) This case corresponds to case (1) of Theorem 2.27. We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$.

There are derivations

$$\psi =_{\parr} \; \rho \, \dfrac{\overset{\star Q}{\overset{\psi' \,\|\, \mathsf{SAKV}^{\downarrow}}{C'}}}{C} \quad , \qquad \overset{\phi_1 \,\|\, \mathsf{SAKV}^{\downarrow}}{A \parr Q} \quad ,$$

with $|\phi_1|_{\parr} \leq |\phi'|_{\parr} < |\phi'|_{\parr}$.

(2) This case corresponds to case (2) of Theorem 2.27. We can apply case 1 of this Theorem 2.37 to ϕ'.

There are derivations

$$\begin{array}{c} H_1 \parr H_2 \\ \psi' \Vert \mathsf{SAKV}^{\downarrow} \\ C_4 \end{array} \quad , \qquad \begin{array}{c} \omega_1 \overline{\Vert} \mathsf{SAKV}^{\downarrow} \\ \star A \parr C_1 \parr H_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} \omega_2 \overline{\Vert} \mathsf{SAKV}^{\downarrow} \\ C_2 \parr C_3 \parr H_2 \end{array} \quad ,$$

with $|\omega_1|_{\parr} + |\omega_2|_{\parr} \leq |\phi'|_{\parr}$.

We apply the induction hypothesis to ω_1 as $|\omega_1|_{\parr} \leq |\phi'|_{\parr} < |\phi|_{\parr}$.

There are derivations

$$\begin{array}{c} \star Q \\ \psi'' \Vert \mathsf{SAKV}^{\downarrow} \\ C_1 + H_1 \end{array} \quad , \qquad \begin{array}{c} \phi_1 \overline{\Vert} \mathsf{SAKV}^{\downarrow} \\ A \parr Q \end{array} \quad ,$$

with $|\phi_1|_{\parr} \leq |\omega_1|_{\parr} < |\phi|_{\parr}$.

We take:

$$\psi =_{\parr} \otimes\!\!\downarrow \frac{\boxed{\begin{array}{c} \star Q \\ \psi'' \Vert \\ C_1 \parr H_1 \end{array}} \otimes \boxed{\begin{array}{c} \omega_2 \overline{\Vert} \\ C_2 \parr C_3 \parr H_2 \end{array}}}{(C_1 \otimes C_3) \parr C_2 \parr \boxed{\begin{array}{c} H_1 \parr H_2 \\ \psi' \Vert \\ C_4 \end{array}}} \quad .$$

(3) This corresponds to case (3) of Theorem 2.27. We can apply case 1 of this Theorem 2.37 to ϕ'. There are derivations

$$\begin{array}{c} H_1 \ \overline{\beta} \ H_2 \\ \psi' \Vert \mathsf{SAKV}^{\downarrow} \\ C_2 \end{array} \quad , \qquad \begin{array}{c} \omega_1 \overline{\Vert} \mathsf{SAKV}^{\downarrow} \\ \star A \parr C_1 \parr H_1 \end{array} \quad , \qquad \begin{array}{c} \omega_2 \overline{\Vert} \mathsf{SAKV}^{\downarrow} \\ u_\beta \parr H_2 \end{array} \quad ,$$

with $|\omega_1|_{\parr} + |\omega_2|_{\parr} \leq |\phi'|_{\parr}$.

By Lemma 2.24, there is a derivation

$$\begin{array}{c} \bar{u}_\beta \\ \psi'' \Vert \mathsf{SAKV}^{\downarrow} \\ H_2 \end{array} \quad .$$

We apply the induction hypothesis to ω_1 as $|\omega_1|_{\parr} \leq |\phi'|_{\parr} < |\phi|_{\parr}$. There are derivations

$$\begin{array}{c} \star Q \\ \psi''' \Vert \mathsf{SAKV}^{\downarrow} \\ C_1 \parr H_1 \end{array} \quad , \qquad \begin{array}{c} \phi_1 \overline{\Vert} \mathsf{SAKV}^{\downarrow} \\ A \parr Q \end{array} \quad ,$$

with $|\phi_1|_{⅋} \leq |\omega_1|_{⅋} < |\phi'|_{⅋}$.

We take:

$$\psi =_{⅋} \begin{array}{c} \star Q \\ {\scriptstyle \psi'''} \| \\ C_1 ⅋ \boxed{\begin{array}{c} H_1 \ \overline{\beta} \ \boxed{\begin{array}{c} \bar{u}_\beta \\ {\scriptstyle \psi''} \| \\ H_2 \end{array}} \\ {\scriptstyle \psi'} \| \\ C_2 \end{array}} \end{array} .$$

(4) This case is analogous to (3).

(5) This corresponds to case (5) of Theorem 2.27. We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} \star Q \\ {\scriptstyle \psi} \| \mathsf{SAKV}^{\downarrow} \\ C \end{array} , \qquad \phi_1 \equiv \begin{array}{c} {\scriptstyle \phi_1'} \,\widetilde{\|}\, \mathsf{SAKV}^{\downarrow} \\ \boxed{\rho \dfrac{A'}{A}} ⅋ Q \end{array} ,$$

with $|\phi_1|_{⅋} = |\phi_1'|_{⅋} + 1 \leq |\phi'|_{⅋} + 1 = |\phi|_{⅋}$.

(6) This corresponds to case (7) of Theorem 2.27. We can apply the induction hypothesis to ϕ' as $|\phi'|_+ < |\phi|_+$. There are derivations

$$\begin{array}{c} \star H \\ {\scriptstyle \psi'} \| \mathsf{SAKV}^{\downarrow} \\ C_2 \end{array} , \qquad \begin{array}{c} {\scriptstyle \phi_1} \,\widetilde{\|}\, \mathsf{SAKV}^{\downarrow} \\ A ⅋ C_1 ⅋ H \end{array} ,$$

with $|\phi_1|_{⅋} \leq |\phi'|_{⅋} < |\phi|_{⅋}$.

We take $Q \equiv C_1 ⅋ H$, and

$$\psi =_{⅋} \star\!\downarrow \frac{\star(C_1 ⅋ H)}{\star C_1 ⅋ \boxed{\begin{array}{c} \star H \\ {\scriptstyle \psi'} \| \\ C_2 \end{array}}} .$$

(7) This corresponds to case (15) of Theorem 2.27. By Lemma 2.24, there is a

derivation $\begin{array}{c}\circ\\ \psi' \Vert \mathsf{SAKV}^{\downarrow}\\ C\end{array}$ and we take:

$$\psi \equiv \begin{array}{c} = \dfrac{\star\circ}{\circ} \\ \psi' \Vert \\ C \end{array} \quad , \qquad \phi_1 \equiv =_{\bindnasrepma} \frac{1}{\circ \bindnasrepma \circ} \quad ,$$

with $|\phi_1|_{\bindnasrepma} = 0 \leq |\phi|_{\bindnasrepma}$.

□

Theorem 2.38. *For any formula A and any context $S\{\ \}$, given a proof $\begin{array}{c}\phi \Vert \mathsf{SAKV}^{\downarrow}\\ S\{A\}\end{array}$, there exist a formula K, a provable context $H\{\ \}$ and derivations*

$$\begin{array}{c}\zeta \Vert \mathsf{SAKV}^{\downarrow}\\ A + K\end{array} \qquad \textit{and} \qquad \begin{array}{c} H\{\{\ \} + K\} \\ \chi \Vert \mathsf{SAKV}^{\downarrow} \\ S\{\ \}\end{array} \ .$$

Proof. We proceed by induction on the number of relations $\alpha \neq \bindnasrepma$ that $\{\ \}$ is in the scope of in $S\{\ \}$. We denote it by $|S|_{\bindnasrepma}$.

If $|S|_+ = 0$, then $S\{A\} =_{\bindnasrepma} A \bindnasrepma K$ and we take $\zeta =_{\bindnasrepma} \phi$ and $H\{\ \} = \{\ \}$.

If $S\{A\} =_{\bindnasrepma} (S'\{A\}\ \beta\ B) \bindnasrepma C$ we proceed as in Theorem 2.29.

If $S\{A\} =_{\bindnasrepma} \star S'\{A\} \bindnasrepma C$, we apply Theorem 2.27 to ϕ. There exist derivations

$$\begin{array}{c}\star Q\\ \psi \Vert \mathsf{SAKV}^{\downarrow}\\ C\end{array} \quad , \quad \begin{array}{c}\phi_1 \Vert \mathsf{SAKV}^{\downarrow}\\ S'\{A\} \bindnasrepma Q\end{array} \ .$$

We apply the induction hypothesis to ϕ_1 since $|S'|_+ < |S|_+$. There are derivations

$$\begin{array}{c}\zeta \Vert \mathsf{SAKV}^{\downarrow}\\ A \bindnasrepma K\end{array} \quad , \quad \begin{array}{c} H'\{\{\ \} \bindnasrepma K\} \\ \chi' \Vert \mathsf{SAKV}^{\downarrow} \\ S'\{\ \} \bindnasrepma Q_1\end{array} \quad ,$$

with H' a provable context.

We take $H\{\ \} = \star H'\{\ \}$. We have $H\{1\} = \star H'\{1\} = \star 1 = \star\circ = \circ = 1$, and we can build in $\mathsf{SAKV}^{\downarrow}$

$$\chi \equiv \star\!\downarrow \frac{\star \boxed{\begin{array}{c} H'\{\{\ \} \parr K\} \\ \chi' \Vert \\ S'\{\ \} \parr Q \end{array}}}{\star S'\{\ \} \parr \boxed{\begin{array}{c} \star Q \\ \psi \Vert \\ C \end{array}}} \quad .$$

$\square$

Elimination of the rules $\parr\!\uparrow$, $a\!\uparrow$, $\triangleleft\!\uparrow$ is a consequence of Theorem 2.31. We will focus on showing the admissibility of the rule $\star\!\uparrow$ in an identical argument, showcasing tha fact that admissibility is a broad phenomenon related to the particular shape of rules and extending beyond the cut.

Corollary 2.39 (Admissibility of $\star\!\uparrow$)**.** *Let* SA *be a splittable proof system.*

For any formulae A, B, C, D, any context S, given a proof

$$\phi \equiv \begin{array}{c} \phi' \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ S\left\{ \alpha\!\uparrow \dfrac{\star A \otimes \star B}{\star(A \otimes B)} \right\} \end{array} \quad ,$$

there is a proof

$$\begin{array}{c} \pi \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ S\{\star(A \otimes B)\} \end{array} \quad .$$

Proof. We apply Theorem 2.38 to ϕ.

There are derivations

$$\begin{array}{c} \zeta \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ (\star A \otimes \star B) \parr K \end{array} \qquad \text{and} \qquad \begin{array}{c} H\{\{\ \} \parr K\} \\ \chi \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ S\{\ \} \end{array} \quad ,$$

with $H\{1\} = 1$.

We apply Theorem 2.37 to ζ. There exist derivations

$$\begin{array}{c} Q_1 \parr Q_2 \\ \psi \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ K \end{array} \quad , \quad \begin{array}{c} \phi_1 \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ \star A \parr Q_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} \phi_2 \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ \star B \parr Q_2 \end{array} \quad .$$

We apply Theorem 2.37 to ϕ_3 and ϕ_4 and we obtain

$$\begin{array}{c} \star Q_A \\ \psi_1 \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ Q_1 \end{array} \quad , \quad \begin{array}{c} \phi_3 \, \Vert \, \mathsf{SAKV}^{\downarrow} \\ Q_A \parr A \end{array} \quad ,$$

$$
\begin{array}{c} \star Q_B \\ \psi_2 \Vert \mathsf{SAKV}^{\downarrow} \\ Q_2 \end{array} \quad , \qquad \begin{array}{c} \phi_4 \,\Vert\, \mathsf{SAKV}^{\downarrow} \\ Q_B \parr B \end{array} \quad ,
$$

We can then build the following proof in $\mathsf{SAKV}^{\downarrow}$

$$
\pi = \begin{array}{c}
H \left\{ \begin{array}{c}
\star\!\!\downarrow \dfrac{\star \boxed{\otimes\!\!\downarrow \dfrac{\boxed{\begin{array}{c}\phi_3 \Vert \\ A \parr Q_A\end{array}} \otimes \boxed{\begin{array}{c}\phi_4 \Vert \\ B \parr Q_B\end{array}}}{(A \otimes B) \parr Q_A \parr Q_B}}}{\star(A \otimes B) \parr \boxed{\begin{array}{c} \star\!\!\downarrow \dfrac{\star(Q_A \parr Q_B)}{\boxed{\begin{array}{c}\star Q_A \\ \psi_1 \Vert \\ Q_1\end{array}} \parr \boxed{\begin{array}{c}\star Q_B \\ \psi_2 \Vert \\ Q_2\end{array}}} \\ \psi \Vert \\ K \end{array}}}
\end{array} \right\} \\
\chi \Vert \\
S\{\star(A \otimes B)\}
\end{array} .
$$

□

2.4 Conclusions

The general splitting procedure gives us a full understanding of *how* the splitting procedure works, and *why* it has been shown to work in every linear system expressed in deep inference so far. We have shown that dualities and the interactions between linear rules are the fundamental phenomena behind admissibility. In this way, we come to see admissibility as a property resulting from the shape of rules that extends beyond the cut: we can show the admissibility of a whole class of inference rules. Furthermore, the understanding that we gain from the generalised theorem allows us to showcase just how broad this methodology is. We have given sufficient properties verified by a whole class of substructural logics that are enough to prove cut-elimination.

Splitting is a global procedure: we have to take into consideration the whole proof to find independent subproofs and rearrange them. This comes only at a polynomial-time complexity cost, and the size of the cut-free proof is at most linear on the size of the original proof. Therefore we see that linear rules do not contribute towards the complexity cost of cut-elimination procedures.

Last, the generalisation of splitting does not only contribute to the understanding of the procedure, it also provides guidelines for the design of logical systems. By providing a generalised theorem, we are able to remove the search for cut-elimination from the design process.

Chapter 3

Decomposition

It is a well known phenomenon in proof theory that in many systems derivations can be arranged into consecutive subderivations made up of only certain rules. For example, we can decompose a first-order proof into a propositional phase and a quantified phase through a Herbrand theorem [9]. This phenomenon has long been explored in deep inference [6, 29, 32, 40, 21], presenting decomposition by means of specific permutations of rules or super-rules, permuting the contractions and cuts together.

Decomposition theorems provide a way to normalise proofs and divide derivations into independent subsystems that can be studied independently. Furthermore, they give the possibility of dividing cut-elimination into several different procedures: decomposition, which introduces complexity, and cut-elimination on a proper linear fragment which does not.

Although decomposition theorems abound, it is the separation of a particular subsystem that we are after: it has long been conjectured that classical logic and linear logic proofs can be decomposed into a splittable phase and a contractive phase independently from cut-elimination, as happens for example in the logic NEL [30] or in the multiplicative exponential fragment of linear logic [40].

In fact, obtaining a total decomposition into a splittable phase followed by a contractive phase is equivalent to showing that general contractions such as the inference

$$\frac{A \vee A}{A}$$

in classical logic can be permuted to the bottom of linear proofs. However, as is pointed out in [40], it is not always clear whether (and how) this general rule permutes with other rules of the system.

The locality awarded by deep inference allows us to advance towards this result, since we can permute atomic contractions to the bottom of a proof in both classical logic [32] and linear logic [40] through reduction rules for proofs. The decomposition procedures that yield these results are independent from cut-elimination in the case of proofs that do not contain a particular type of subderivation, called a *cycle*.

The decomposition results in for atomic contractions in the literature that we will present in the next subsection are a significant step towards proving these conjectures, but need to be expanded in two ways to obtain a full decomposition result independent from cut-elimination. The first one is that for both classical logic and linear logic cut-elimination is used to prove the termination of the decomposition procedure, to show that cycles can be removed from proofs. The second one is that it is unclear how rules involved in making contractions atomic, such as the rule m of SKS, should be permuted with other rules.

In this chapter we will present general reduction rules for systems that achieve four goals:

- We are able to show that the existing decomposition results for classical logic and linear logic are obtained via reductions that are in fact instances of a more general reduction coming from the interactions of contractive rules with other rules;
- We present sufficient conditions for two rules to permute with each other, reducing the analysis usually necessary to obtain decomposition results;
- We show that decomposition and cut-elimination are independent procedures by providing a local procedure to remove cycles through these reduction rules;
- We present tools for future work on achieving a full decomposition theorem for both classical logic and linear logic.

These results fundamentally exploit the regularity of the rules in subatomic systems, reducing the study of the permutation of rules to only two cases.

We will start by introducing the reduction rules given in [32] to obtain the decomposition result for atomic contractions in classical logic. We will introduce *atomic flows*, an invariant of proofs that allows us to intuitively follow these reductions and the measure used to prove the termination of the reduction system in the absence of cycles. Following that, we will present a generalisation of the notion of contraction, and characterise a type of rules, called *contractive*, which we can permute downwards in a proofs through the general reduction rules we present. In the last chapter we will use these generalised reduction rules to present a procedure allowing us to remove cycles from proofs without recurring to cut-elimination.

3.1 Preliminaries: atomic decomposition in classical logic and multiplicative additive linear logic

In system SKS (Figure 1-2) it is possible to obtain reduction rules to permute atomic contractions $ac{\downarrow}$ and atomic cocontractions $ac{\uparrow}$ towards the bottom or the top of a derivation respectively. We will introduce the rewriting system for derivations presented in [32] to achieve that.

Definition 3.1. A *reduction rule* r is a couple (ϕ', ψ') where ϕ' and ψ' are derivations in SKS with $\mathsf{pr}\,\phi' \equiv \mathsf{pr}\,\psi'$ and $\mathsf{cn}\,\phi' \equiv \mathsf{cn}\,\psi'$. We write $r : \phi' \to \psi'$.

For every reduction rule $r : \phi' \to \psi'$ we define the reduction $\to_r$ such that $\phi \to_r \psi$ if and only if ψ' is a subderivation of ϕ and ψ is obtained from ϕ by replacing ϕ' by ψ'.

We call a finite set R of reduction rules a *rewriting system*. Given a set S of derivations, we say that rewriting system R is *terminating on S* if there is no infinite chain $\phi \to_{r_1} \phi_1 \to_{r_2} \ldots$ with $r_i \in R$ for any $\phi \in S$.

Definition 3.2. We define the following reduction rules for SKS:

- $c\!\downarrow\!-c\!\downarrow$:

$$ {\scriptstyle ac\uparrow}\dfrac{{\scriptstyle ac\downarrow}\dfrac{a \vee a}{a}}{a \wedge a} \quad\longrightarrow\quad {\scriptstyle m}\dfrac{\boxed{{\scriptstyle ac\uparrow}\dfrac{a}{a \wedge a}} \vee \boxed{{\scriptstyle ac\uparrow}\dfrac{a}{a \wedge a}}}{\boxed{{\scriptstyle ac\downarrow}\dfrac{a \vee a}{a}} \wedge \boxed{{\scriptstyle ac\downarrow}\dfrac{a \vee a}{a}}} $$

- $c\!\downarrow\!-i\!\uparrow$:

$$ {\scriptstyle ai\uparrow}\dfrac{\boxed{{\scriptstyle ac\downarrow}\dfrac{a \vee a}{a}} \wedge \bar{a}}{\mathsf{f}} \quad\longrightarrow\quad {\scriptstyle =}\dfrac{{\scriptstyle s}\dfrac{{\scriptstyle s}\dfrac{(a \vee a) \wedge \boxed{{\scriptstyle ac\uparrow}\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}}{(a \wedge (\bar{a} \wedge \bar{a})) \vee a}}{\boxed{{\scriptstyle ai\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{{\scriptstyle ai\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}}}}}{\mathsf{f}} $$

- $c\!\downarrow\!-w\!\uparrow$:

$$ {\scriptstyle aw\uparrow}\dfrac{{\scriptstyle ac\downarrow}\dfrac{a \vee a}{a}}{\mathsf{t}} \quad\longrightarrow\quad {\scriptstyle =}\dfrac{\boxed{{\scriptstyle w\uparrow}\dfrac{a}{\mathsf{t}}} \vee \boxed{{\scriptstyle w\uparrow}\dfrac{a}{\mathsf{t}}}}{\mathsf{t}} $$

And their duals:

- $i\!\downarrow\!-c\!\uparrow$:

$$ {\scriptstyle ai\downarrow}\dfrac{\mathsf{t}}{\boxed{{\scriptstyle ac\uparrow}\dfrac{a}{a \wedge a}} \vee \bar{a}} \quad\longrightarrow\quad {\scriptstyle s}\dfrac{{\scriptstyle s}\dfrac{{\scriptstyle =}\dfrac{\mathsf{t}}{\boxed{{\scriptstyle ac\downarrow}\dfrac{\mathsf{t}}{a \vee \bar{a}}} \wedge \boxed{{\scriptstyle ac\downarrow}\dfrac{\mathsf{t}}{a \vee \bar{a}}}}}{((a \vee \bar{a}) \wedge a) \vee \bar{a}}}{(a \wedge a) \vee \boxed{{\scriptstyle ac\downarrow}\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}}} $$

- $w{\downarrow}-c{\uparrow}$:

$$\dfrac{\dfrac{\mathsf{f}}{a}\,aw{\downarrow}}{a \wedge a}\,ac{\uparrow} \quad \longrightarrow \quad = \dfrac{\mathsf{f}}{\boxed{\dfrac{\mathsf{f}}{a}\,aw{\downarrow}} \wedge \boxed{\dfrac{\mathsf{f}}{a}\,aw{\downarrow}}}$$

Last, we define the trivial family of reduction rules:

- $c{\downarrow}-\rho_H$:

$$\rho\,\dfrac{H\left\{ ac{\downarrow}\,\dfrac{a \vee a}{a} \right\}}{H'\{a\}} \quad \longrightarrow \quad \rho\,\dfrac{H\{a \vee a\}}{H'\left\{ ac{\downarrow}\,\dfrac{a \vee a}{a} \right\}}$$

- $\rho_H-c{\uparrow}$:

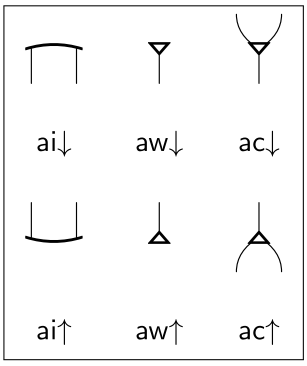
$$\rho\,\dfrac{H'\{a\}}{H\left\{ ac{\uparrow}\,\dfrac{a}{a \wedge a} \right\}} \quad \longrightarrow \quad \rho\,\dfrac{H'\left\{ ac{\uparrow}\,\dfrac{a}{a \wedge a} \right\}}{H\{a \wedge a\}}$$

It is clear that if the rewriting system obtained from the reduction rules of definition 3.1 terminates, then we will obtain a derivation with three phases: a top phase made up only of rules $ac{\uparrow}$, a phase made-up of rules s,m, $ai{\uparrow}$, $ai{\downarrow}$, $w{\uparrow}$, $w{\downarrow}$ and a bottom phase made up only of rules $ac{\downarrow}$.

Definition 3.3. We define *rewriting system* C for SKS as the rewriting system given by the reduction rules of Definition 3.1.

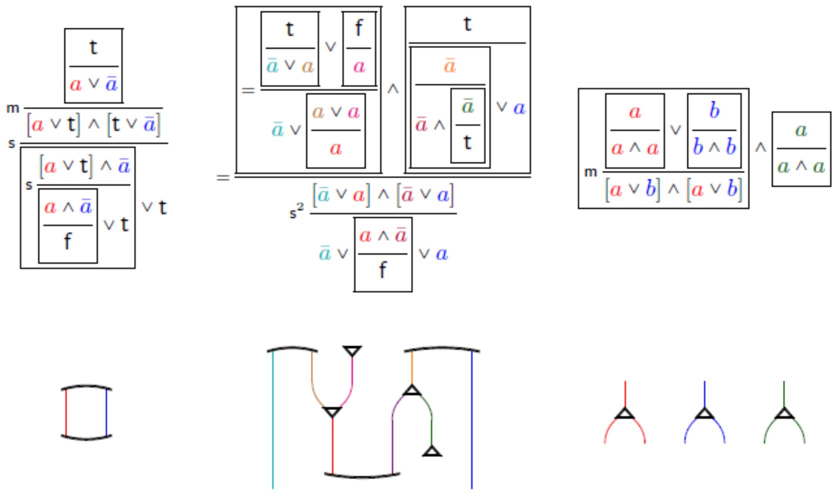
We will see that in the absence of a certain construction inside a derivation, called cycle, the termination of rewriting system C is guaranteed. To provide a measure for termination, we will introduce the *atomic flows*, a graphical invariant of proofs that allows us to intuitively follow these reductions.

Atomic flows are specialised Buss flow graphs [10] that follow the occurrences of atoms in a derivation in SKS. They can be seen as composite diagrams that are freely generated from a set of six elementary diagrams, or as labeled directed graphs, where the six possible labels for the vertices are given in the following figure.



We can associate an atomic flow to every derivation in SKS in a natural way: every edge follows the occurrence of an atom in the derivation, and each vertex label corresponds to the occurrence of a critical rule where atoms are created or destroyed ($ai{\downarrow}, ai{\uparrow}, aw{\downarrow}, aw{\uparrow}, ac{\downarrow}, ac{\uparrow}$). The direction of the edges corresponds to the up-down direction in a derivation. The units f and t are not represented in the flow.

Example 3.4. Below are several examples of derivations and the flows associated to them. Every edge represents an occurrence of the atom of the same colour.



Technically, there are some restrictions on the construction of the flows to guarantee that for every flow there is an associated SKS derivation. However, only an intuitive understanding of the flows is required to follow the graphical representation of the rewriting rules and the measure presented in this section and this is what we are seeking to provide. The interested reader is invited to refer to [32] for further details on the definition of the atomic flows and on the definitions and results presented in what follows.

The measure used to prove termination can be easily followed in a flow: it corresponds to the length of a certain type of paths.

Definition 3.5. Given an edge ϵ in an atomic flow, we define $\mathsf{up}(\epsilon)$ as the upper vertex it is connected to, and $\mathsf{lo}(\epsilon)$ as the lower vertex it is connected to.

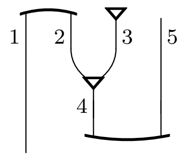
Given a sequence of distinct edges $\epsilon_1, \dots, \epsilon_n$ such that $\mathsf{lo}(\epsilon_i) = \mathsf{up}(\epsilon_{i+1})$ for $1 \leq i < n$, we say that $\epsilon_1, \dots, \epsilon_n$ is a *path of length n from* $\mathsf{up}(\epsilon_1)$ *to* $\mathsf{lo}(\epsilon_n)$, and that $\epsilon_n, \dots, \epsilon_1$ is a *path of length n from* $\mathsf{lo}(\epsilon_n)$ *to* $\mathsf{up}(\epsilon_1)$.

Given a sequence of distinct edges $\epsilon_1, \dots, \epsilon_n$, we say that $\epsilon_1, \dots, \epsilon_n$ is an *ai-path of length n from vertex v_1 to vertex v_2* if it is a path from v_1 to v_2 or if there exists a vertex v labeled by $ai{\uparrow}$ or $ai{\downarrow}$ such that $\epsilon_1, \dots, \epsilon_h$ is an ai-path from v_1 to v and $\epsilon_{h+1}, \dots, \epsilon_n$ is an ai-path from v to v_2.

An ai-path of length n is *maximal* if no ai-path containing its edges has length greater than n. An ai-path of length n from v is *maximal* if no ai-path from v containing its edges has length greater than n.

Intuitively, paths correspond to any non-empty sequence of edges from v_1 to v_2 that does not change direction (it either only 'goes downwards' or 'goes upwards'). ai-paths are allowed to change direction, but only at ai-vertices: they are zig-zag paths that change direction at ai-nodes.

Example 3.6.



Some examples of paths of this flow are $2, 4$ and 5.
Some examples of ai-paths in this flow are given by $1, 2$ and $3, 4, 5$.
The maximal ai-paths of this flow are $1, 2, 4, 5$ and $3, 4, 5$ and their reverse.
The maximal ai-paths from the $ac\downarrow$ vertex are $2, 1$ and 3 and $4, 5$.

If we consider the maximal ai-paths from an $ac\downarrow$ vertex starting with its lower edge, we can see that their length corresponds to the number of critical rules the contraction it corresponds to will have to "go through" when applying the reduction rules. For example, in a derivation whose flow is the flow of example 3.6, when we apply the reduction rules to move the atomic contraction downwards, it will permute with one instance of the rule $ai\uparrow$.

More precisely, we can assign a *rank* to every contraction and to every cocontraction of a derivation by refering to its flow. The rank of a contraction will be given by the sum of the lengths of the maximal ai-paths starting with the lower edge of its corresponding vertex in the flow. Dually, the rank of a cocontraction will be given by the sum of the lengths of the maximal ai-paths starting with the upper edge of its corresponding vertex in a flow. We will see that the reduction rules of system C reduce the sum of the ranks of the contractions and cocontractions in a derivation, effectively providing a termination measure when these ranks are finite.

Definition 3.7. Given a vertex v labelled with $ac\downarrow$ in a flow, we define its *rank* as the sum of the lengths of the maximal ai-paths $\epsilon_1, \ldots, \epsilon_n$ from v such that $\mathsf{up}(\epsilon_1) = v$.

Dually, given a vertex v labelled with $ac\uparrow$ in a flow, we define its *rank* as the sum of the lengths of the maximal ai-paths $\epsilon_1, \ldots, \epsilon_n$ from v such that $\mathsf{lo}(\epsilon_1) = v$.

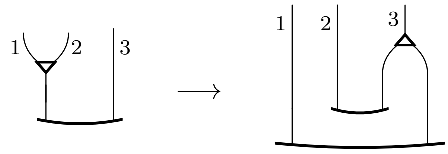
Example 3.8. The rank of the $ac\downarrow$ vertex of the flow of example 3.6 is 2: it corresponds to the length of the ai-path $4, 5$.

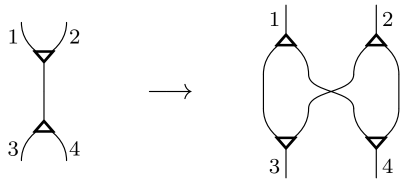
Definition 3.9. Given an occurrence of the rule $ac\downarrow$ in a derivation ϕ with flow ψ, we define its *rank* as the rank of its corresponding vertex in ψ.

Likewise, we define the rank of an occurrence of the rule $ac\uparrow$ as the rank of its corresponding vertex.

The reductions of system C will reduce the sum of the ranks of the contractions and cocontractions in a derivation except when a certain construction is present, that we call an *ai*-cycle.

This can perhaps best be seen by considering the atomic flow reductions associated to the reductions on derivations:

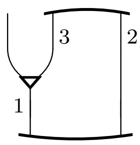
$c{\downarrow}-i{\uparrow}$: 

$c{\downarrow}-c{\uparrow}$: 

It is easy to check that the sum of the ranks of $ac{\downarrow}$ and $ac{\uparrow}$ vertexes is decreased by these reductions, when the cycles defined in what follows are not present.

Definition 3.10. An *ai*-path from v to v is called an *ai-cycle*.

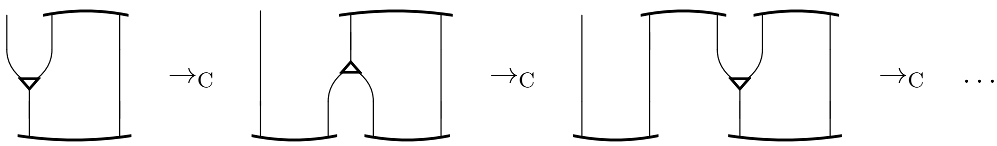
Example 3.11.



The *ai*-path $1, 2, 3$ is an *ai*-cycle.

Definition 3.12. We say that a derivation contains an *ai*-cycle if its atomic flow contains an *ai*-cycle.

When we apply the reductions in C to atomic contractions that belong to a cycle, the rewriting system is not terminating:



In the absence of *ai*-cycles however, the rewriting system terminates as is proved in [32]. We simply outline that proof here to give the reader an idea of the proof and to show that the termination measure and arguments can easily be extended to the rewriting system for MALL that we will present next.

Theorem 3.13. *Rewriting system* C *is terminating on the set of ai-cycle-free derivations.*

Proof. The first observation is that it is clear by inspection of the reduction rules that the rank of (co)contractions not involved in the reduction stays the same.

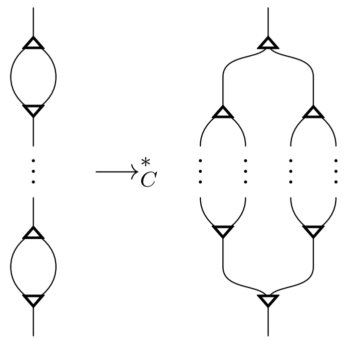
Given an ai-cycle-free derivation ϕ, we consider the lexicographic order on (r, d). r is the sum of the ranks of the contractions and cocontractions in ϕ, and d is the sum of the number of rules below each contraction and the number of rules above each cocontraction when sequentialising ϕ.

We will show that each application of a reduction of C reduces (r, d).

- Applications of the rules $c\!\downarrow\!-c\!\uparrow$, $c\!\downarrow\!-i\!\uparrow$ and $i\!\downarrow\!-c\!\uparrow$ reduce r in the absence of ai-cycles as is shown in the proof of Theorem 7.2.3 of [32] .
- Applications of the rules $c\downarrow -w\uparrow$ and $w\downarrow -c\uparrow$ reduce r since they remove contractions and cocontractions.
- Applications of the rules $c\!\downarrow\!-\rho_H$ and $\rho_H\!-c\!\uparrow$ trivially maintain r and reduce d.

∎

The decomposition procedure may increase the size of a proof exponentially, through the crossings of contractions and cocontractions in the following configuration:



The formula corresponding to the middle line of the diagram on the right will contain a number of atoms exponentially larger than any of the formulae corresponding to the diagram on the left.

This poses a stark contrast with the polynomial cost of cut-elimination via splitting: by separating the two procedures we are able to isolate the source of the complexity cost of cut-elimination in cycle-free proofs.

ai-cycles are evidently removed through cut-elimination, since they are caused by the connexion of a cut and an introduction. In Chapter 4 we will present a procedure to remove loops that does not involve cut-elimination, thus proving the independence of decomposition from cut-elimination. The complexity cost of that procedure is as of yet unknown, and is the last missing element in understanding and separating the causes of the complexity cost of cut-elimination.

Weakenings and coweakenings can be permuted to the bottom/top of a derivation easily through the following reductions, presented in [32] as well.

Definition 3.14. We define the following reduction rules for SKS:

- $w\downarrow - c\downarrow$:

$$\dfrac{\boxed{aw\downarrow\dfrac{\mathsf{f}}{a}} \vee a}{a}\,ac\downarrow \quad \longrightarrow \quad = \dfrac{\mathsf{f} \vee a}{a}$$

- $w\downarrow - i\uparrow$:

$$\dfrac{\boxed{aw\downarrow\dfrac{\mathsf{f}}{a}} \wedge \bar{a}}{\mathsf{f}}\,ai\uparrow \quad \longrightarrow \quad = \dfrac{\mathsf{f} \wedge \boxed{aw\uparrow\dfrac{\bar{a}}{\mathsf{t}}}}{\mathsf{f}}$$

- $w\downarrow - w\uparrow$:

$$aw\uparrow\dfrac{aw\downarrow\dfrac{\mathsf{f}}{a}}{\mathsf{t}} \quad \longrightarrow \quad = \dfrac{s\dfrac{=\dfrac{\mathsf{f}}{\mathsf{f} \wedge (\mathsf{f} \vee \mathsf{t})}}{(\mathsf{f} \wedge \mathsf{f}) \vee \mathsf{t}}}{\mathsf{t}}$$

And their duals:

- $c\uparrow - w\uparrow$:

$$ac\uparrow\dfrac{a}{\boxed{aw\uparrow\dfrac{a}{\mathsf{t}}}} \quad \longrightarrow \quad = \dfrac{a}{a \wedge \mathsf{t}}$$

- $i\downarrow - w\uparrow$:

$$ai\downarrow\dfrac{\mathsf{t}}{\boxed{aw\uparrow\dfrac{a}{\mathsf{t}}} \vee \bar{a}} \quad \longrightarrow \quad = \dfrac{\mathsf{t}}{\mathsf{t} \vee \boxed{aw\downarrow\dfrac{\mathsf{f}}{\bar{a}}}}$$

And the trivial reductions:

- $w\downarrow - \rho_H$:

$$\rho\,\dfrac{H\left\{aw\downarrow\dfrac{\mathsf{f}}{a}\right\}}{H'\{\mathsf{f}\}} \quad \longrightarrow \quad \rho\,\dfrac{H\{\mathsf{f}\}}{H'\left\{aw\downarrow\dfrac{\mathsf{f}}{a}\right\}}$$

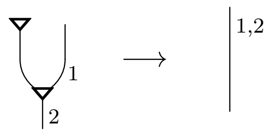
- $\rho_H - w\uparrow$:

$$\rho\,\frac{H'\{a\}}{H\left\{aw\uparrow\frac{a}{\mathsf{t}}\right\}} \quad\longrightarrow\quad \rho\,\frac{H'\left\{aw\uparrow\frac{a}{\mathsf{t}}\right\}}{H\{\mathsf{t}\}}$$

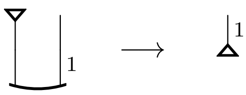
Definition 3.15. We define rewriting system W as the rewriting system given by the reductions in Definition 3.14.

By observing the corresponding flow reductions, it is easy to see that the non-trivial reductions of W remove edges of atomic flows:

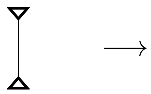
$w\downarrow - c\downarrow$:



$w\downarrow - i\uparrow$:



$w\downarrow - u\uparrow$:



Termination is then clear, since every application of a non-trivial reduction rule reduces the number of edges of the associated flow to a derivation, and the trivial rules reduce the number of rules below weakenings and above coweakenings. By a similar argument to the one used for Theorem 3.13, we will then obatin termination.

Theorem 3.16. *Rewriting system* W *is terminating.*

Note that the reductions of system W do not introduce atomic (co)contractions or medials: only splittable rules. By applying system C followed by system W to a derivation, we obtain an SKS derivation of the form

$$\begin{array}{l} A \\ \Vert\, w\uparrow \\ A_1 \\ \Vert\, ac\uparrow \\ A_2 \\ \Vert\, s,m,ai \\ A_3 \\ \Vert\, ac\downarrow \\ A_4 \\ \Vert\, w\downarrow \\ B \end{array} \quad .$$

$$
ai\!\downarrow \frac{1}{a \parr \bar{a}} \qquad\qquad ai\!\uparrow \frac{a \otimes \bar{a}}{\bot}
$$

$$
s\,\frac{(A \parr B) \otimes C}{(A \otimes C) \parr B}
$$

$$
d\!\downarrow \frac{(A \parr B) \mathbin{\&} (C \parr D)}{(A \mathbin{\&} C) \parr (B \oplus D)} \qquad d\!\uparrow \frac{(A \oplus B) \otimes (C \mathbin{\&} D)}{(A \otimes C) \oplus (B \otimes D)}
$$

$$
\oplus\!\downarrow \frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)} \qquad \&\!\uparrow \frac{(A \mathbin{\&} B) \otimes (C \mathbin{\&} D)}{(A \otimes C) \mathbin{\&} (B \otimes D)}
$$

$$
m\,\frac{(A \mathbin{\&} B) \oplus (C \mathbin{\&} D)}{(A \oplus C) \mathbin{\&} (B \oplus D)}
$$

$$
ac\!\downarrow \frac{a \oplus a}{a} \qquad\qquad ac\!\uparrow \frac{a}{a \mathbin{\&} a}
$$

$$
m_2\!\downarrow \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)} \qquad m_2\!\uparrow \frac{(A \mathbin{\&} B) \parr (C \mathbin{\&} D)}{(A \parr C) \mathbin{\&} (B \parr D)}
$$

$$
at\!\downarrow \frac{0}{a} \qquad\qquad at\!\uparrow \frac{a}{\top}
$$

Figure 3-1: *System* SMALLS

Extremely similar rewriting systems can be presented for linear logic [40] to permute atomic (co)contractions with the other rules. We will particularly focus on the multiplicative additive fragment of linear logic (MALL) given by the subsystem SMALLS (Figure 3-1) corresponding to the MALL fragment of the system SLLS in [40]. The exponentials are expected to be included in future research as unary relations.

We will briefly introduce the rewriting systems, to highlight the similarities between the reduction rules in classical logic and in linear logic, and to observe that an identical termination argument than that made for Theorem 3.13 holds for derivations without ai-cycles in multiplicative additive linear logic.

Definition 3.17. We present the following reduction rules for SMALLS:

- $c\!\downarrow - c\!\downarrow$:

$$
\begin{array}{c} ac\!\downarrow \dfrac{a \oplus a}{\,ac\!\uparrow \dfrac{a}{a \mathbin{\&} a}\,} \end{array}
\quad \longrightarrow \quad
m\,\frac{\boxed{ac\!\uparrow \dfrac{a}{a \mathbin{\&} a}} \oplus \boxed{ac\!\uparrow \dfrac{a}{a \mathbin{\&} a}}}{\boxed{ac\!\downarrow \dfrac{a \oplus a}{a}} \mathbin{\&} \boxed{ac\!\downarrow \dfrac{a \oplus a}{a}}}
$$

- $c{\downarrow}-i{\uparrow}$:

$$ai{\uparrow}\,\dfrac{\boxed{ac{\downarrow}\,\dfrac{a \oplus a}{a}} \otimes \bar{a}}{\bot} \quad \longrightarrow \quad =\dfrac{d{\uparrow}\,\dfrac{(a \oplus a) \otimes \boxed{ac{\uparrow}\,\dfrac{\bar{a}}{\bar{a} \,\&\, \bar{a}}}}{\boxed{ai{\uparrow}\,\dfrac{a \,\&\, \bar{a}}{\bot}} \oplus \boxed{ai{\uparrow}\,\dfrac{a \,\&\, \bar{a}}{\bot}}}}{\bot}$$

- $c{\downarrow}-w{\uparrow}$:

$$aw{\uparrow}\,\dfrac{ac{\downarrow}\,\dfrac{a \oplus a}{a}}{\top} \quad \longrightarrow \quad =\dfrac{\boxed{w{\uparrow}\,\dfrac{a}{\top}} \oplus \boxed{w{\uparrow}\,\dfrac{a}{\top}}}{\top}$$

Just like for classical logic, we can define the duals of these reductions and the trivial reduction rules.

Definition 3.18. Rewriting system Q for SMALLS is given by the reduction rules presented in Definition 3.17 and their duals.

We can define the rank of atomic contractions and atomic cocontractions in an identical fashion to classical logic, and present the exact same argument for the termination of Q in the absence of *ai*-cycles.

Theorem 3.19. *Rewriting system* Q *is terminating on the set of ai-cycle-free* SMALLS *derivations.*

Again, this decomposition procedure may increase the size of a proof exponentially, through the exact same phenomenon as in classical logic.

We can define reduction rules for the permutation of weakenings and coweakenings.

Definition 3.20. We define the following reduction rules for SMALLS:

- $w{\downarrow}-c{\downarrow}$:

$$ac{\downarrow}\,\dfrac{\boxed{aw{\downarrow}\,\dfrac{\mathsf{0}}{a}} \oplus a}{a} \quad \longrightarrow \quad =\dfrac{\mathsf{0} \oplus a}{a}$$

- $w{\downarrow}-i{\uparrow}$:

$$ai{\uparrow}\,\dfrac{\boxed{aw{\downarrow}\,\dfrac{\mathsf{0}}{a}} \otimes \bar{a}}{\bot} \quad \longrightarrow \quad =\dfrac{\mathsf{0} \otimes \boxed{aw{\uparrow}\,\dfrac{\bar{a}}{\top}}}{\bot}$$

- $w{\downarrow}-w{\uparrow}$:

$$
\dfrac{\dfrac{0}{a}\,aw{\downarrow}}{\top}\,aw{\uparrow}
\quad\longrightarrow\quad
\dfrac{\dfrac{\dfrac{0}{(\bot \parr \top) \,\&\, (\bot \parr 0)}\,=}{(\bot \,\&\, \bot) \oplus (\top \oplus 0)}\,d{\downarrow}}{\top}\,=
$$

We can define the dual reductions and the trivial reductions identically to clasical logic.

Definition 3.21. Rewriting system Y for SMALLS is given by the reduction rules of Definition 3.20 together with their duals and the trivial reduction rules.

Just like for classical logic, these reduction rules remove atoms from a derivation. Therefore, the rewriting system is clearly terminating.

Theorem 3.22. *Rewriting system Y is terminating.*

Again, we can remark that the reductions of system Y do not introduce atomic (co)contractions or other contractive rules: only splittable rules $d{\downarrow}$ and $d{\uparrow}$.

We have thus shown that it is possible to decompose SKS and SMALLS derivations in extremely similar ways. In the next section we will show that both decomposition theorems correspond to the same phenomenon: the interaction of contractive rules. Furthermore, in the last section of this chapter we will present a procedure to remove ai-cycles from derivations, effectively showing the independence of decomposition and cut-elimination.

3.2 General rewriting system

Decomposition theorems obtained by permutations of rules, being a local phenomenon, are as different as different logics are. Therefore, generalising decomposition is not a straightforward task. However, permuting atomic contractions to the bottom of a proof has been proved possible in both classical logic and in linear logic (Section 3.1). The reduction rules to achieve it are extremely similar in both logics, suggesting that they are heavily dependant on the shape of the rules rather than being system-specific.

Furthermore, it has long been a conjecture that it is possible to further decompose proofs into a splittable phase followed by the other rules in classical logic [6] and in linear logic, suggesting that we can permute rules other than atomic contractions downwards in a proof as well.

Both these arguments indicate that it should be possible to characterise the rules that can be permuted downwards in proofs and generalise the reduction rules. This is what we set out to do in this section: we will present generalised reduction rules that encompass the existing reduction rules for classical logic and linear logic, as well as allow us to permute other contractive rules downwards in a proof. It is expected that future research will yield a full decomposition theorem for classical logic by means of these reductions.

In addition, these reduction rules will be fundamental in the *ai*-cycle removal procedure that we will present in Chapter 4.

The main problem we face when permuting contractive rules such as the rule m of SKS downwards in a proof is that it is not clear how to proceed, since by permuting it through certain rules we may create an unbounded number of cocontractions and medials, making it extremely difficult to guarantee that we are in fact advancing towards a medial-free proof and to find a measure that will show the termination of the procedure.

By observing the subatomic reduction rules corresponding to the reductions presented in the previous section, a novel way of controlling this phenomenon arises: we will show that it is possible to move 'blocks' of nested contractive rules together, in such a way that we are no longer concerned by the number of cocontractions and medials created by the decomposition procedure.

The reduction $c{\downarrow} \to c{\uparrow}$ for SKS can for example be written subatomically as

$$a\bar{c}\,\dfrac{\boxed{ac\,\dfrac{((\mathsf{f}\wedge\mathsf{f})\;a\;(\mathsf{t}\wedge\mathsf{t}))\vee((\mathsf{f}\wedge\mathsf{f})\;a\;(\mathsf{t}\wedge\mathsf{t}))}{\boxed{\wedge c\,\dfrac{((\mathsf{f}\wedge\mathsf{f})\vee(\mathsf{f}\wedge\mathsf{f}))}{\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\wedge\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}}}\;a\;\boxed{\wedge c\,\dfrac{((\mathsf{t}\wedge\mathsf{t})\vee(\mathsf{t}\wedge\mathsf{t}))}{\boxed{\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}\wedge\boxed{\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}}}}{(\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{f}\;a\;\mathsf{t})}
\quad\longrightarrow\quad
\wedge c\,\dfrac{\boxed{a\bar{c}\,\dfrac{(\mathsf{f}\wedge\mathsf{f})\;a\;(\mathsf{t}\wedge\mathsf{t})}{(\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{f}\;a\;\mathsf{t})}}\vee\boxed{a\bar{c}\,\dfrac{(\mathsf{f}\wedge\mathsf{f})\;a\;(\mathsf{t}\wedge\mathsf{t})}{(\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{f}\;a\;\mathsf{t})}}}{\boxed{ac\,\dfrac{(\mathsf{f}\;a\;\mathsf{t})\vee(\mathsf{f}\;a\;\mathsf{t})}{\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\;a\;\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}}}\wedge\boxed{ac\,\dfrac{(\mathsf{f}\;a\;\mathsf{t})\vee(\mathsf{f}\;a\;\mathsf{t})}{\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\;a\;\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}}}}$$

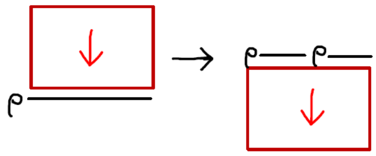
This reduction corresponds to moving a block of nested contractions (in red) by creating another block of nested contractions lower in the proof.

The rule $c{\downarrow}-i{\uparrow}$ can be written subatomically as

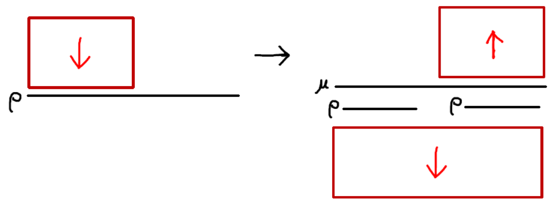
$$a{\uparrow}\,\dfrac{\boxed{ac\,\dfrac{(\mathsf{f}\;a\;\mathsf{t})\vee(\mathsf{f}\;a\;\mathsf{t})}{\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\;a\;\boxed{\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}\wedge(\mathsf{t}\;a\;\mathsf{f})}{(\mathsf{f}\wedge\mathsf{t})\;a\;(\mathsf{t}\wedge\mathsf{f})}
\quad\longrightarrow\quad
\wedge{\uparrow}\,\dfrac{((\mathsf{f}\;a\;\mathsf{t})\vee(\mathsf{f}\;a\;\mathsf{t}))\wedge\boxed{a\bar{c}\,\dfrac{\boxed{\dfrac{\mathsf{t}}{\mathsf{t}\wedge\mathsf{t}}}\;a\;\boxed{\dfrac{\mathsf{f}}{\mathsf{f}\wedge\mathsf{f}}}}{(\mathsf{t}\;a\;\mathsf{f})\wedge(\mathsf{t}\;a\;\mathsf{f})}}}{\boxed{ac\,\dfrac{\boxed{a{\uparrow}\,\dfrac{(\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{t}\;a\;\mathsf{f})}{(\mathsf{f}\wedge\mathsf{t})\;a\;(\mathsf{t}\wedge\mathsf{f})}}\vee\boxed{a{\uparrow}\,\dfrac{(\mathsf{f}\;a\;\mathsf{t})\wedge(\mathsf{t}\;a\;\mathsf{f})}{(\mathsf{f}\wedge\mathsf{t})\;a\;(\mathsf{t}\wedge\mathsf{f})}}}{\boxed{\wedge c\,\dfrac{(\mathsf{f}\wedge\mathsf{t})\vee(\mathsf{f}\wedge\mathsf{t})}{\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\wedge\boxed{\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}\;a\;\boxed{\wedge c\,\dfrac{(\mathsf{t}\wedge\mathsf{f})\vee(\mathsf{t}\wedge\mathsf{f})}{\boxed{\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}\wedge\boxed{\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}}}}}}$$

In this case we move a block of nested contractions by creating another block of nested contractions lower in the proof and a block of nested cocontractions.

We will study these blocks of nested contractions, that we name *merge contractions*. We will show that, by only having a single rule shape to consider, only two cases of non-trivial permutations require our attention. As it turns out, merge contractions permute with other rules in a fashion that mimics the behaviour of atomic contractions. We will present two types of reductions, corresponding to the two types of reductions that we have just shown as examples: a reduction s given by

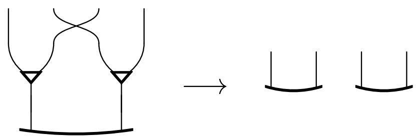


and a reduction t given by



This newly defined structure will give us novel reductions for derivations, such as the reduction

$$\mathsf{ai}{\uparrow}\,\frac{\boxed{m\,\dfrac{(a\wedge\bar a)\vee(a\wedge\bar a)}{\boxed{\mathsf{ac}\,\dfrac{a\vee a}{a}}\wedge\boxed{\mathsf{ac}\,\dfrac{\bar a\vee\bar a}{\bar a}}}}}{\mathsf{f}} \;\to\; =\,\frac{\boxed{\mathsf{ai}{\uparrow}\,\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee\boxed{\mathsf{ai}{\uparrow}\,\dfrac{a\wedge\bar a}{\mathsf{f}}}}{\mathsf{f}}$$



that is fundamental for the cycle-elimination procedure that we will present in the next chapter.

In this section we will use classical logic and multiplicative additive linear logic as examples. However, instead of taking associativity and commutativity as equality axioms, we will present them as instances of rules $\dfrac{(A\ \alpha\ B)\ \alpha\ (C\ \alpha\ D)}{(A\ \alpha\ C)\ \alpha\ (B\ \alpha\ D)}$ (Figures 3-2 and 3-3). This small change does not warrant a change of name for the system, and therefore we will refer to this system for classical logic as SAKS as well.

Definition 3.23 (System $\mathsf{SAMALLS}$)**.** Subatomic formulae for multiplicative additive linear logic $\mathscr{F}$ are given by the set of constants $\mathscr{U} = \{\bot, 0, \top, 1\}$ and the set of relations $\mathscr{R} = \{⅋, \oplus, \&, \otimes\} \cup \mathscr{A}$ where $\mathscr{A}$ is a denumerable set of atoms, denoted by a, b, ... Two examples of subatomic formulae for linear logic are

$$C \equiv ((1 \parr \bot) \; a \; 1) \otimes 0 \quad \text{and} \quad D \equiv ((0 \mathbin{\&} \top) \; b \; 1) \; a \; (1 \parr \bot) \quad .$$

For the set of subatomic formulae for linear logic $\mathscr{F}$, we define negation through:

$$\begin{aligned}
&\bar{\otimes} = \parr \\
&\bar{\&} := \oplus \; ; \\
&\bar{a} := a \text{ for all} a \in \mathscr{A} \; ; \\
&\bar{1} := \bot \; ; \\
&\bar{\top} := 0 \; .
\end{aligned}$$

We define the equational theory $=$ on $\mathscr{F}$ as the minimal equivalence relation closed under negation and under context defined by:

$$\forall A, B, C \in \mathscr{F},$$

$$\begin{array}{ll}
A \otimes 1 = A \; ; & A \parr \bot = A \; ; \\
A \mathbin{\&} \top = A \; ; & A \oplus 0 = A \; ; \\
\\
\bot \mathbin{\&} \bot = \bot \; ; & 1 \mathbin{\&} 1 = 1 \; ; \\
\bot \oplus \bot = \bot \; ; & 1 \oplus 1 = 1 \; ; \\
0 \otimes 0 = 0 \; ; & \top \parr \top = \top \; ; \\
0 \parr 0 = 0 \; ; & \top \otimes \top = \top \; ; \\
0 \mathbin{\&} 0 = 0 \; ; & \top \oplus \top = \top \; ; \\
\\
\forall a \in \mathscr{A}. \; \bot \; a \; \bot = \bot \; ; & \forall a \in \mathscr{A}. \; 1 \; a \; 1 = 1 \; ; \\
\forall a \in \mathscr{A}. \; 0 \; a \; 0 = 0 \; ; & \forall a \in \mathscr{A}. \; \top \; a \; \top = \top \; ; \\
\forall a \in \mathscr{A}. \; \bot \; a \; \top = \top \; ; & \forall a \in \mathscr{A}. \; 1 \; a \; 0 = 0 \; ; \\
\forall a \in \mathscr{A}. \; \top \; a \; \bot = \top \; ; & \forall a \in \mathscr{A}. \; 0 \; a \; 1 = 0 \; ; \\
\forall a \in \mathscr{A}. \; 1 \; a \; \top = \top \; ; & \forall a \in \mathscr{A}. \; \bot \; a \; 0 = 0 \; ; \\
\forall a \in \mathscr{A}. \; \top \; a \; 1 = \top \; ; & \forall a \in \mathscr{A}. \; 0 \; a \; \bot = 0 \; ;
\end{array}$$

A natural interpretation is given by considering the assignments:

$- \ I(1) \equiv 1$;	$- \ I(\bot) \equiv \bot$;
$- \ I(\top) \equiv \top$;	$- \ I(0) \equiv 0$;
$- \ \forall a \in \mathscr{A}.\ I(\bot\, a\, \bot) \equiv \bot$;	$- \ \forall a \in \mathscr{A}.\ I(1\, a\, 1) \equiv 1$;
$- \ \forall a \in \mathscr{A}.\ I(\bot\, a\, 1) \equiv a$;	$- \ \forall a \in \mathscr{A}.\ I(1\, a\, \bot) \equiv \bar{a}$;
$- \ \forall a \in \mathscr{A}.\ I(0\, a\, 0) \equiv 0$;	$- \ \forall a \in \mathscr{A}.\ I(\top\, a\, \top) \equiv \top$;
$- \ \forall a \in \mathscr{A}.\ I(\bot\, a\, \top) \equiv \top$;	$- \ \forall a \in \mathscr{A}.\ I(\top\, a\, \bot) \equiv \top$;
$- \ \forall a \in \mathscr{A}.\ I(\top\, a\, 1) \equiv \top$;	$- \ \forall a \in \mathscr{A}.\ I(1\, a\, \top) \equiv \top$;
$- \ \forall a \in \mathscr{A}.\ I(0\, a\, 1) \equiv 0$;	$- \ \forall a \in \mathscr{A}.\ I(1\, a\, 0) \equiv 0$;
$- \ \forall a \in \mathscr{A}.\ I(\bot\, a\, 0) \equiv 0$;	$- \ \forall a \in \mathscr{A}.\ I(0\, a\, \bot) \equiv 0$;
$- \ I(A \parr B) \equiv I(A) \parr I(B)$;	$- \ I(A \otimes B) \equiv I(A) \otimes I(B)$;
$- \ I(A \oplus B) \equiv I(A) \oplus I(B)$;	$- \ I(A \mathbin{\&} B) \equiv I(A) \mathbin{\&} I(B)$.

where $A, B \in \mathscr{F}^i$, extending it in such a way that $A\, a\, B$ is interpretable iff $A = u, B = v$ with $u, v \in \{\bot, 0, \top, 1\}$ and $u\, a\, v$ is interpretable. Then, $I(A\, a\, B) \equiv I(u\, a\, v)$.

System SAMALLS for multiplicative additive linear logic is given by the inference rules of Figure 3-3 together with an equality rule for each pair of formulae on opposite sides of an equality in the equations above.

System SAMALLS is correct for the multiplicative additive fragment of system SLLS in [39]. Every rule of that fragment trivially corresponds to a rule of SAMALLS, except for the rules $at{\downarrow}$ and $at{\uparrow}$ that are obtained identically to the rules $aw{\downarrow}$ and $aw{\uparrow}$ of classical logic in example 1.41.

The first step in the generalisation is to characterise the *contractions*, the rules that will be permuted. Unsurprisingly, the rules that we will be able to permute downwards/upwards in a derivation correspond to the rules involved in making contraction atomic. We will call them *contractions* as well.

ν-contractive systems will then be defined in such a way that they correspond to those systems where we can always recover *general contractions* of the form

$$\frac{A\ \nu\ A}{A} \quad .$$

Definition 3.24. Let ν be a relation with unit $\triangledown$, and $\overline{\nu}$ its dual with unit $\triangle$. A *ν-contractive system* SA is a subatomic proof system where:

- For every relation α there is a down rule of the form

$$\alpha c\,\frac{(A\ \alpha\ B)\ \nu\ (C\ \alpha\ D)}{(A\ \nu\ C)\ \alpha\ (B\ \nu\ D)} \quad ,$$

  that we call *contraction for α*.

$$
a{\downarrow}\,\frac{(A \vee B)\, a\, (C \vee D)}{(A\, a\, C) \vee (B\, a\, D)} \qquad a{\uparrow}\,\frac{(A\, a\, B) \wedge (C\, a\, D)}{(A \wedge C)\, a\, (B \wedge D)}
$$

$$
\wedge{\downarrow}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \vee{\uparrow}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}
$$

$$
\vee{\downarrow}\,\frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)} \qquad \wedge{\uparrow}\,\frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}
$$

$$
m\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}
$$

$$
ac\,\frac{(A\, a\, B) \vee (C\, a\, D)}{(A \vee C)\, a\, (B \vee D)} \qquad a\bar{c}\,\frac{(A \wedge B)\, a\, (C \wedge D)}{(A\, a\, C) \wedge (B\, a\, D)}
$$

Figure 3-2: SAKS

$$
a{\downarrow}\,\frac{(A \parr B)\, a\, (C \parr D)}{(A\, a\, C) \parr (B\, a\, D)} \qquad a{\uparrow}\,\frac{(A\, a\, B) \otimes (C\, a\, D)}{(A \otimes C)\, a\, (B \otimes D)}
$$

$$
\otimes{\downarrow}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr{\uparrow}\,\frac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)}
$$

$$
\&{\downarrow}\,\frac{(A \parr B)\, \&\, (C \parr D)}{(A\, \&\, C) \parr (B \oplus D)} \qquad \oplus{\uparrow}\,\frac{(A \oplus B) \otimes (C\, \&\, D)}{(A \otimes C) \oplus (B \otimes D)}
$$

$$
\oplus{\downarrow}\,\frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)} \qquad \&{\uparrow}\,\frac{(A\, \&\, B) \otimes (C\, \&\, D)}{(A \otimes C)\, \&\, (B \otimes D)}
$$

$$
\parr{\downarrow}\,\frac{(A \parr B) \parr (C \parr D)}{(A \parr C) \parr (B \parr D)} \qquad \otimes{\uparrow}\,\frac{(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}
$$

$$
m\,\frac{(A\, \&\, B) \oplus (C\, \&\, D)}{(A \oplus C)\, \&\, (B \oplus D)}
$$

$$
ac\,\frac{(A\, a\, B) \oplus (C\, a\, D)}{(A \oplus C)\, a\, (B \oplus D)} \qquad a\bar{c}\,\frac{(A\, \&\, B)\, a\, (C\, \&\, D)}{(A\, a\, C)\, \&\, (B\, a\, D)}
$$

$$
\otimes c\,\frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)} \qquad \parr\bar{c}\,\frac{(A\, \&\, B) \parr (C\, \&\, D)}{(A \parr C)\, \&\, (B \parr D)}
$$

$$
\oplus c\,\frac{(A \oplus B) \oplus (C \oplus D)}{(A \oplus C) \oplus (B \oplus D)} \qquad \&\bar{c}\,\frac{(A\, \&\, B)\, \&\, (C\, \&\, D)}{(A\, \&\, C)\, \&\, (B\, \&\, D)}
$$

Figure 3-3: SAMALLS

- Dually, for every relation α there is an up-rule of the form

$$\alpha\bar{c}\,\frac{(A\;\overline{\nu}\;B)\;\alpha\;(C\;\overline{\nu}\;D)}{(A\;\alpha\;B)\;\overline{\nu}\;(C\;\alpha\;D)}\quad,$$

  that we call *cocontraction for α.*

- For every constant $u \in \mathcal{U}$ there is a unit assignment for ν of the form $u\;\nu\;u = u$. We call the equality rule $=\dfrac{u\;\nu\;u}{u}$ the *contraction equality rule for u.*

- Dually, for every constant $u \in \mathcal{U}$ there is a unit assignment for $\overline{\nu}$ of the form $u\;\overline{\nu}\;u = u$. We call the equality rule $=\dfrac{u}{u\;\overline{\nu}\;u}$ the *cocontraction equality rule for u.*

- For every constant $u \in \mathcal{U}$, $\dfrac{\triangledown}{u}$ is derivable in SA. We will denote these unitary instances of contraction rules by $w\,\dfrac{\triangledown}{u}$ and call them *weakenings.*

- Dually, for every constant $u \in \mathcal{U}$, $\dfrac{u}{\vartriangle}$ is derivable in SA. We will denote these unitary instances of contraction rules by $\bar{w}\,\dfrac{u}{\vartriangle}$ and call them *coweakenings.*

- For every relation α there is an equality axiom $\triangledown\;\alpha\;\triangledown = \triangledown$.

- Dually, for every relation α there is an equality axiom $\vartriangle\alpha\vartriangle=\vartriangle$.

We call ν the *contracting relation*, and $\overline{\nu}$ the *cocontracting relation.*

Remark 3.25. Note that this definition implies that ν is weak.

Example 3.26. System SAKS (Figure 3-2) is a $\vee$-contractive system.

Example 3.27. System $\mathsf{SAMALLS}$ (Figure 3-3) is a $\oplus$-contractive system.

The structure that we are interested in studying is that of nestings of contraction rules, just like the blocks we highlighted in the introductory example. It is these blocks that we will show it is possible to permute downwards in a proof. For convenience and readability, we will represent these nestings in the form of a hyper-rule $\mathsf{mc}{\downarrow}\,\dfrac{A\;\nu\;B}{C}$, named *merge contraction*, which will be defined recursively in order to capture the nested structure.

Definition 3.28. In a ν-contractive system SA, *a nesting of contractions* is an SA derivation defined recursively as follows:

- A formula $A \ \nu \ B$ is a nesting of contractions ;
- A contraction equality rule is a nesting of contractions ;
- A derivation

$$c \frac{(A \ \alpha \ B) \ \nu \ (C \ \alpha \ D)}{\boxed{\begin{array}{c} A \ \nu \ C \\ \phi_1 \, \| \\ R \end{array}} \ \alpha \ \boxed{\begin{array}{c} B \ \nu \ D \\ \phi_2 \, \| \\ S \end{array}}}$$

  is a nesting of contractions if c is a contraction and ϕ_1 and ϕ_2 are nestings of contractions.

Nestings of cocontractions are defined dually.

Definition 3.29. A *ν-merge* of two formulae is defined as follows:

- $A \ \nu \ B$ is a ν-merge of A and B that we call a *trivial merge*;
- u is a ν-merge of u and u, where $u \in \mathscr{U}$ is a constant;
- $C_1 \ \alpha \ C_2$ is a ν-merge of $A_1 \ \alpha \ A_2$ and $B_1 \ \alpha \ B_2$ for $\alpha \in \mathscr{R}$ if C_1 is a ν-merge of A_1 and B_1 and C_2 is a ν-merge of A_2 and B_2.

If C is a ν-merge of A and B, by an abuse of language we will sometimes refer to the triple (A, B, C) as a ν-merge.

$\overline{\nu}$-merges of two formulae are likewise defined as follows:

- $A \ \overline{\nu} \ B$ is a $\overline{\nu}$-merge of A and B that we call a *trivial merge*;
- u is a $\overline{\nu}$-merge of u and u, where $u \in \mathscr{U}$ is a constant;
- $C_1 \ \alpha \ C_2$ is a $\overline{\nu}$-merge of $A_1 \ \alpha \ A_2$ and $B_1 \ \alpha \ B_2$ for $\alpha \in \mathscr{R}$ if C_1 is a $\overline{\nu}$-merge of A_1 and B_1 and C_2 is a $\overline{\nu}$-merge of A_2 and B_2.

Note that the merge of two formulae is not unique.

Proposition 3.30. *Given a nesting of contractions* $\begin{array}{c} A \ \nu \ B \\ \phi \, \| \\ C \end{array}$ *, C is a ν-merge of A and B.*

Dually, given a nesting of cocontractions $\begin{array}{c} C \\ \psi \, \| \\ A \ \overline{\nu} \ B \end{array}$ *, C is a $\overline{\nu}$-merge of A and B.*

Proof. We proceed by induction on the length of ϕ.

- If $\phi \equiv A \ \nu \ B$, it is clear.

- If ϕ is a contraction equality rule, it is clear.
- If $\phi \equiv {}_{c}\dfrac{(A \;\alpha\; B) \;\nu\; (C \;\alpha\; D)}{\boxed{\begin{array}{c} A \;\nu\; C \\ \phi_1 \,\| \\ R \end{array}} \;\alpha\; \boxed{\begin{array}{c} B \;\nu\; D \\ \phi_2 \,\| \\ S \end{array}}}$ with c a contraction and ϕ_1 and ϕ_2 nestings of contractions, then by induction hypothesis R is a ν-merge of A and C and S is a ν-merge of B and D, and therefore $R \;\alpha\; S$ is a ν-merge of $(A \;\alpha\; B)$ and $(C \;\alpha\; D)$.

We prove the dual likewise. $\square$

Proposition 3.31. *Let* SA *be a ν-contractive system. If C is a ν-merge of A and B, there is a nesting of contractions* $\begin{array}{c} A \;\nu\; B \\ \| \\ C \end{array}$.

Dually, If C is a $\overline{\nu}$-merge of A and B, there is a nesting of cocontractions $\begin{array}{c} C \\ \| \\ A \;\overline{\nu}\; B \end{array}$.

Proof. We proceed by structural induction on C:

- if $C \equiv A \;\nu\; B$, we take $\phi \equiv A \;\nu\; B$;
- if $C \equiv A \equiv B \equiv u$ with $u \in \mathscr{U}$, we take $\phi \equiv {}_{=}\dfrac{u \;\nu\; u}{u}$;
- if $C \equiv C_1 \;\alpha\; C_2$, $A \equiv A_1 \;\alpha\; A_2$, $B \equiv B_1 \;\alpha\; B_2$ where $\alpha \in \mathscr{R}$ and C_1 is a ν-merge of A_1 and B_1 and C_2 is a ν-merge of A_2 and B_2, we take

$$\phi \equiv {}_{\alpha c}\dfrac{(A_1 \;\alpha\; A_2) \;\nu\; (B_1 \;\alpha\; B_2)}{\boxed{\begin{array}{c} A_1 \;\nu\; B_1 \\ \phi_1 \,\| \\ C_1 \end{array}} \;\alpha\; \boxed{\begin{array}{c} A_2 \;\nu\; B_2 \\ \phi_2 \,\| \\ C_2 \end{array}}}$$

  where ϕ_1 and ϕ_2 are the nestings of contractions associated to the ν-merges (A_1, B_1, C_1) and (A_2, B_2, C_2) respectively obtained by the induction hypothesis.

We prove the dual likewise. $\square$

Given the above characterisation of nestings as derivations whose conclusion is a ν-merge of its premiss, for ease of notation we will represent nestings as a hyper-rule, that we call merge contraction.

Definition 3.32. In a ν-contractive system SA,

${}_{\mathsf{mc}\downarrow}\dfrac{A \;\nu\; B}{C}$ is a *merge contraction* if C is a non-trivial ν-merge of A and B;

${}_{\mathsf{mc}\uparrow}\dfrac{C}{A \;\overline{\nu}\; B}$ is a *merge cocontraction* if C is a non-trivial $\overline{\nu}$-merge of A and B.

If C is of the form $C_1 \ \alpha \ C_2$, we say that the merge (co)contraction has *main relation* α.

Clearly, all the contraction rules of SA are merge-contractions, and all cocontraction rules are merge cocontractions.

For each nesting, we have a merge contraction, and for each merge contraction we have a nesting. We will permute nestings downwards in a derivation by creating other nestings lower in the derivation. We will show this by, equivalently, permuting merge contractions downwards by creating other merge contractions lower in the derivation.

General contractions $\dfrac{A \ \nu \ A}{A}$ are a particular case of merge contractions. The following proposition can be proved by an obvious structural induction on the formula.

Proposition 3.33. *For any formula A there is a merge contraction*

$$\mathsf{mc}\!\downarrow \frac{A \ \nu \ A}{A} \quad .$$

Dually there is a merge cocontraction

$$\mathsf{mc}\!\uparrow \frac{A}{A \ \overline{\nu} \ A} \quad .$$

The fact that we can consider merge contractions (or nestings) as a single block is an important contribution of the reduction rules presented in what follows: reduction rules may introduce an unbounded number of cocontraction rules, which are an issue in the search of a measure to prove the termination of a full decomposition procedure. By considering them as a single block however, we greatly simplify this search.

In contractive systems where formulae are built over the units of relations, weakenings come 'for free'. This is a consequence of the fact that the inferences $w \dfrac{\triangledown}{u_\alpha}$ are always derivable in a ν-contractive system. If u_α is a unit for α, then we can consider the following instance of a contractive inference rule:

$$\alpha c \frac{(u_\alpha \ \alpha \ \triangledown) \ \nu \ (\triangledown \ \alpha \ u_\alpha)}{(u_\alpha \ \nu \ \triangledown) \ \alpha \ (\triangledown \ \nu \ u_\alpha)}$$

with premiss $\triangledown$ and conclusion u_α.

Through these unitary weakenings and the equations $\triangledown \ \alpha \ \triangledown = \triangledown$, we can recover general weakenings

$$\frac{\triangledown}{A}$$

as well.

In fact, we will give weakenings a special treatment rules, and will therefore not permute them downwards in a proof with the reductions presented in what follows. We will instead present different reduction rules for them, as is done for the weakenings in the previous section.

Lemma 3.34. *In a ν-contractive system, for every formula A there is a derivation*

$$\begin{array}{c} \triangledown \\ \phi \,\Vert\, \{=,w\} \\ A \end{array}$$

made-up only of weakenings and equalities. By an abuse of language, we will call it weakening.

Proof. We proceed by structural induction on A.

If $A \equiv u$, then we take $w \dfrac{\triangledown}{u}$.

If $A \equiv A_1 \ \alpha \ A_2$, then by induction hypothesis there are derivations

$$\begin{array}{c} \triangledown \\ \phi_1 \,\Vert\, \{=,aw\} \\ A_1 \end{array} \quad \text{and} \quad \begin{array}{c} \triangledown \\ \phi_2 \,\Vert\, \{=,w\} \\ A_2 \end{array} \quad .$$

We take

$$\phi \equiv \; = \dfrac{\triangledown}{\boxed{\begin{array}{c} \triangledown \\ \phi_1 \,\Vert\, \{=,w\} \\ A_1 \end{array}} \ \alpha \ \boxed{\begin{array}{c} \triangledown \\ \phi_2 \,\Vert\, \{=,w\} \\ A_2 \end{array}}} \quad .$$

□

The following definition presents an important property of merge contractions that will allow us to permute them with other rules.

Definition 3.35. If C is a ν-merge of A and B, we define the *projections* $\begin{array}{c} A \\ \pi_A \,\Vert\, \{=,w\} \\ C \end{array}$ *and* $\begin{array}{c} B \\ \pi_B \,\Vert\, \{=,w\} \\ C \end{array}$ *associated to the merge* recursively as follows:

- if $C \equiv A \ \nu \ B$, we take $\pi_A = \; = \dfrac{A}{A \ \nu \ \boxed{\begin{array}{c} \triangledown \\ \Vert\, \{=,w\} \\ B \end{array}}}$ and $\pi_B \equiv \; = \dfrac{B}{\boxed{\begin{array}{c} \triangledown \\ \Vert\, \{=,w\} \\ A \end{array}} \ \nu \ B}$;

- if $C \equiv A \equiv B \equiv u$ with $u \in \mathcal{U}$, we take $\pi_A \equiv u$ and $\pi_B \equiv u$;
- if $C \equiv C_1 \; \alpha \; C_2$, $A \equiv A_1 \; \alpha \; A_2$, $B \equiv B_1 \; \alpha \; B_2$ where $\alpha \in \mathcal{R}$ and C_1 is an ν-merge of A_1 and B_1 and C_2 is an ν-merge of A_2 and B_2, we take

$$\pi_A \equiv \begin{array}{|c|}\hline A_1 \\ \pi_{A_1} \parallel \{=,w\} \\ C_1 \\ \hline\end{array} \; \alpha \; \begin{array}{|c|}\hline A_2 \\ \pi_{A_2} \parallel \{=,w\} \\ C_2 \\ \hline\end{array} \qquad \text{and} \qquad \pi_B \equiv \begin{array}{|c|}\hline B_1 \\ \pi_{B_1} \parallel \{=,w\} \\ C_1 \\ \hline\end{array} \; \alpha \; \begin{array}{|c|}\hline B_2 \\ \pi_{B_2} \parallel \{=,w\} \\ C_2 \\ \hline\end{array} \quad ,$$

  where π_{A_1}, π_{B_1} are the projections associated to the merge (A_1, B_1, C_1) and π_{A_2}, π_{B_2} are the projections associated to the merge (A_2, B_2, C_2).

With the projections associated to a merge as a tool, we will now show reduction rulles allowing us to permute merge (co)contractions downwards (upwards) in a proof.

Notation 3.36. We will write

$$\frac{(A \; \beta \; B) \; \gamma \; (C \; \beta' \; D)}{(A \; \gamma \; C) \; \beta \; (B \; \gamma' \; D)}$$

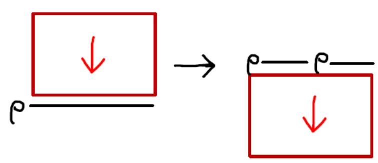
to represent both up and down-rules, i.e. either $\beta'{=}\beta$ and $\gamma'{=}\gamma^m$ or $\beta'{=}\beta^M$ and $\gamma'{=}\gamma$.

Definition 3.37. A *subatomic reduction rule* r for a system SA is a couple (ϕ', ψ') where ϕ' and ψ' are derivations in SA with $\mathsf{pr}\,\phi' \equiv \mathsf{pr}\,\psi'$ and $\mathsf{cn}\,\phi' \equiv \mathsf{cn}\,\psi'$. We write $r : \phi' \to \psi'$.

For every reduction rule $r : \phi' \to \psi'$ we define the reduction $\to_r$ such that $\phi \to_r \psi$ if and only if ψ' is a subderivation of ϕ and ψ is obtained from ϕ by replacing ϕ' by ψ'.

We call a finite set R of reduction rules a *rewriting system*. Given a set S of SA derivations, we say that rewriting system R is *weakly normalising on* S if for every $\phi \in S$ there is a finite chain $\phi \to_{r_1} \phi_1 \to_{r_2} \cdots \to_{r_n} \psi$ with $r_i \in R$ where no reduction rule of R can be applied to ψ.

The first family of reduction rules we present is akin to the rule $c\!\uparrow - c\!\downarrow$ for atomic flows.



Merge contractions permute with a rule directly below them by duplicating it.

Definition 3.38 (Reduction rule s)**.** In a ν-contractive system, we define the following class of reduction rules:

$$
s_\rho : \quad \mathsf{mc}{\downarrow}\,\frac{A \;\nu\; B}{C\left\{\rho\,\dfrac{M}{N}\right\}} \quad\longrightarrow\quad \mathsf{mc}{\downarrow}\,\frac{\boxed{\begin{array}{c}A\\ \pi_A \,\|\\ C\left\{\rho\,\dfrac{M}{N}\right\}\end{array}} \;\nu\; \boxed{\begin{array}{c}B\\ \pi_B \,\|\\ C\left\{\rho\,\dfrac{M}{N}\right\}\end{array}}}{C\{N\}}
$$

where π_A and π_B are the projections associated to the merge (A, B, C).

Since these projections exist for any merge, this rewriting always holds.

Example 3.39. The reduction rule $c{\uparrow}-c{\downarrow}$ for atomic flows is an instance of this reduction rule. Likewise, the reduction rule presented in [40] to permute atomic contractions and atomic cocontractions in linear logic is an instance of this reduction rule family:

$$
\mathsf{mc}{\uparrow}\,\frac{\mathsf{mc}{\downarrow}\,\dfrac{(\bot\, a\, 1) \oplus (\bot\, a\, 1)}{\bot\, a\, 1}}{(\bot\, a\, 1) \mathbin{\&} (\bot\, a\, 1)} \quad\longrightarrow\quad \mathsf{mc}{\downarrow}\,\frac{\boxed{\mathsf{mc}{\uparrow}\,\dfrac{\bot\, a\, 1}{(\bot\, a\, 1) \mathbin{\&} (\bot\, a\, 1)}} \oplus \boxed{\mathsf{mc}{\uparrow}\,\dfrac{\bot\, a\, 1}{(\bot\, a\, 1) \mathbin{\&} (\bot\, a\, 1)}}}{(\bot\, a\, 1) \mathbin{\&} (\bot\, a\, 1)}
$$

or, written in terms of nestings:

$$
a\bar{c}\,\frac{\boxed{ac\,\dfrac{((\bot \mathbin{\&} \bot)\, a\, (1 \mathbin{\&} 1)) \oplus ((\bot \mathbin{\&} \bot)\, a\, (1 \mathbin{\&} 1))}{\boxed{\mathbin{\&}c\,\dfrac{((\bot \mathbin{\&} \bot) \oplus (\bot \mathbin{\&} \bot))}{\boxed{\dfrac{\bot \oplus \bot}{\bot}} \mathbin{\&} \boxed{\dfrac{\bot \oplus \bot}{\bot}}}} \; a \; \boxed{\mathbin{\&}c\,\dfrac{((1 \mathbin{\&} 1) \oplus (1 \mathbin{\&} 1))}{\boxed{\dfrac{1 \oplus 1}{1}} \mathbin{\&} \boxed{\dfrac{1 \oplus 1}{1}}}}}}}{(\bot\, a\, 1) \mathbin{\&} (\bot\, a\, 1)} \quad\longrightarrow\quad \mathbin{\&}c\,\frac{\boxed{a\bar{c}\,\dfrac{(\bot \mathbin{\&} \bot)\, a\, (1 \mathbin{\&} 1)}{(\bot\, a\, 1) \mathbin{\&} (\bot\, a\, 1)}} \oplus \boxed{a\bar{c}\,\dfrac{(\bot \mathbin{\&} \bot)\, a\, (1 \mathbin{\&} 1)}{(\bot\, a\, 1) \mathbin{\&} (\bot\, a\, 1)}}}{\boxed{ac\,\dfrac{(\bot\, a\, 1) \oplus (\bot\, a\, 1)}{\boxed{\dfrac{\bot \oplus \bot}{\bot}} \; a \; \boxed{\dfrac{1 \oplus 1}{1}}}} \mathbin{\&} \boxed{ac\,\dfrac{(\bot\, a\, 1) \oplus (\bot\, a\, 1)}{\boxed{\dfrac{\bot \oplus \bot}{\bot}} \; a \; \boxed{\dfrac{1 \oplus 1}{1}}}}}
$$

Example 3.40. We can apply an instance of this reduction rule to permute rule $\wedge c$ and rule $\wedge{\downarrow}$ of SAKS, i.e. to permute a medial through a switch:

$$
\wedge{\downarrow}\,\frac{\wedge c\,\dfrac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}}{(A \wedge B) \vee (C \vee D)} \quad\longrightarrow\quad \mathsf{mc}{\downarrow}\,\frac{\boxed{\wedge{\downarrow}\,\dfrac{\left(A \vee \begin{array}{c}\triangledown\\ \| \{=,w\}\\ C\end{array}\right) \wedge \left(B \vee \begin{array}{c}\triangledown\\ \| \{=,w\}\\ D\end{array}\right)}{(A \wedge B) \vee (C \vee D)}} \vee \boxed{\wedge{\downarrow}\,\dfrac{\left(\begin{array}{c}\triangledown\\ \| \{=,w\}\\ A\end{array} \vee C\right) \wedge \left(\begin{array}{c}\triangledown\\ \| \{=,w\}\\ B\end{array} \vee D\right)}{(A \wedge B) \vee (C \vee D)}}}{(A \wedge B) \vee (C \vee D)} \;.
$$

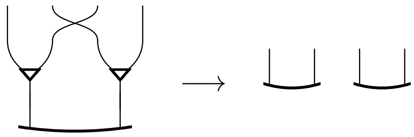
Example 3.41. We can also permute a merge contraction through a cut for example:

$$
a{\uparrow}\,\frac{\mathsf{mc}{\downarrow}\,\dfrac{((\mathsf{f}\, a\, \mathsf{t}) \wedge (\mathsf{t}\, a\, \mathsf{f})) \vee ((\mathsf{f}\, a\, \mathsf{t}) \wedge (\mathsf{t}\, a\, \mathsf{f}))}{(\mathsf{f}\, a\, \mathsf{t}) \wedge (\mathsf{t}\, a\, \mathsf{f})}}{(\mathsf{f} \wedge \mathsf{t})\, a\, (\mathsf{t} \wedge \mathsf{f})} \quad\longrightarrow\quad \mathsf{mc}{\downarrow}\,\frac{\boxed{a{\uparrow}\,\dfrac{(\mathsf{f}\, a\, \mathsf{t}) \wedge (\mathsf{t}\, a\, \mathsf{f})}{(\mathsf{f} \wedge \mathsf{t})\, a\, (\mathsf{t} \wedge \mathsf{f})}} \vee \boxed{a{\uparrow}\,\dfrac{(\mathsf{f}\, a\, \mathsf{t}) \wedge (\mathsf{t}\, a\, \mathsf{f})}{(\mathsf{f} \wedge \mathsf{t})\, a\, (\mathsf{t} \wedge \mathsf{f})}}}{(\mathsf{f} \wedge \mathsf{t})\, a\, (\mathsf{t} \wedge \mathsf{f})}
$$

or, written in terms of nestings:

$$
a\uparrow\dfrac{\wedge c\,\dfrac{((\mathsf{f}\,a\,\mathsf{t})\wedge(\mathsf{t}\,a\,\mathsf{f}))\vee((\mathsf{f}\,a\,\mathsf{t})\wedge(\mathsf{t}\,a\,\mathsf{f}))}{\boxed{ac\,\dfrac{(\mathsf{f}\,a\,\mathsf{t})\vee(\mathsf{f}\,a\,\mathsf{t})}{\boxed{=\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\;a\;\boxed{=\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}\wedge\boxed{ac\,\dfrac{(\mathsf{t}\,a\,\mathsf{f})\vee(\mathsf{t}\,a\,\mathsf{f})}{\boxed{=\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}\;a\;\boxed{=\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}}}}}{(\mathsf{f}\wedge\mathsf{t})\,a\,(\mathsf{t}\wedge\mathsf{f})}
\quad\longrightarrow\quad
ac\,\dfrac{\boxed{a\uparrow\dfrac{(\mathsf{f}\,a\,\mathsf{t})\wedge(\mathsf{t}\,a\,\mathsf{f})}{(\mathsf{f}\wedge\mathsf{t})\,a\,(\mathsf{t}\wedge\mathsf{f})}}\vee\boxed{a\uparrow\dfrac{(\mathsf{f}\,a\,\mathsf{t})\wedge(\mathsf{t}\,a\,\mathsf{f})}{(\mathsf{f}\wedge\mathsf{t})\,a\,(\mathsf{t}\wedge\mathsf{f})}}}{\boxed{\wedge c\,\dfrac{(\mathsf{f}\wedge\mathsf{t})\vee(\mathsf{f}\wedge\mathsf{t})}{\boxed{=\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}\wedge\boxed{=\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}}}\;a\;\boxed{\wedge c\,\dfrac{(\mathsf{t}\wedge\mathsf{f})\vee(\mathsf{t}\wedge\mathsf{f})}{\boxed{=\dfrac{\mathsf{t}\vee\mathsf{t}}{\mathsf{t}}}\wedge\boxed{=\dfrac{\mathsf{f}\vee\mathsf{f}}{\mathsf{f}}}}}}
$$

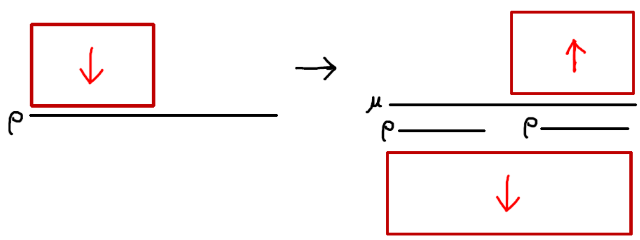
We obtain the flow transformation:



This transformation is a fundamental advance allowing us to remove ai-cycles as we will show in the next chapter. This discovery has been made purely through the means of the subatomic methodology, and it suggests that by studying the behaviour of contractive rules in the same way that atomic flows study the behaviour of atomic contractions we can discover and characterise interesting properties of proof systems.

It is in the case where a generic contraction is "broken" by another rule where it has until now been unclear how to proceed. Just like in the reduction rule $c\downarrow - i\uparrow$, we might create cocontractions, but in this case we might obtain an arbitrarily big number of them.

The main contribution of these reduction rules is the fact that we can now consider all the cocontractions created as a single merge cocontraction block that we can move as a whole upwards in a proof, therefore not having to be concerned by its size.



Unlike for the previous reduction rules, the following rule is not always applicable. However, we can easily present sufficient conditions for its applicability, therefore characterising systems in which merge contractions permute with every other rule.

Definition 3.42 (Reduction rule t)**.** If the rule ${\scriptstyle\mu}\dfrac{(A\ \nu\ B)\ \beta\ (C\ \overline{\nu}\ D)}{(A\ \beta\ C)\ \nu\ (B\ \beta\ D)}$ is derivable in ν-contractive system SA we define the following family of rewriting rules:

$$t_\rho : \quad {\scriptstyle\rho}\dfrac{\boxed{{\scriptstyle\mathsf{mc}\downarrow}\dfrac{(A_1\ \alpha\ A_2)\ \nu\ (B_1\ \alpha\ B_2)}{C\ \alpha\ D}}\ \beta\ (E\ \alpha'\ F)}{(C\ \beta\ E)\ \alpha\ (D\ \beta'\ F)} \quad \longrightarrow$$

$${\scriptstyle\mathsf{mc}\downarrow}\dfrac{{\scriptstyle\mu}\dfrac{((A_1\ \alpha\ A_2)\ \nu\ (B_1\ \alpha\ B_2))\ \beta\ \boxed{{\scriptstyle\mathsf{mc}\uparrow}\dfrac{E\ \alpha'\ F}{(E\ \alpha'\ F)\ \overline{\nu}\ (E\ \alpha'\ F)}}}{\boxed{{\scriptstyle\rho}\dfrac{(A_1\ \alpha\ A_2)\ \beta\ (E\ \alpha'\ F)}{(A_1\ \beta\ E)\ \alpha\ (A_2\ \beta'\ F)}}\ \nu\ \boxed{{\scriptstyle\rho}\dfrac{(B_1\ \alpha\ B_2)\ \beta\ (E\ \alpha'\ F)}{(B_1\ \beta\ E)\ \alpha\ (B_2\ \beta'\ F)}}}}{(C\ \beta\ E)\ \alpha\ (D\ \beta'\ F)}$$

where C is a ν-merge of A_1 and B_1, and D is a ν-merge of A_2 and B_2.

Example 3.43. The reduction rule $c{\downarrow}{-}i{\uparrow}$ for classical logic is an instance of this reduction rule. Likewise, the reduction rule presented in [40] to permute atomic contractions and atomic cuts in linear logic is an instance of this reduction rule family:

$${\scriptstyle a\uparrow}\dfrac{\boxed{{\scriptstyle\mathsf{mc}\downarrow}\dfrac{(\bot\ a\ 1)\oplus(\bot\ a\ 1)}{\bot\ a\ 1}}\otimes(1\ a\ \bot)}{(\bot\otimes 1)\ a\ (1\otimes\bot)} \longrightarrow {\scriptstyle\mathsf{mc}\downarrow}\dfrac{{\scriptstyle\oplus\uparrow}\dfrac{((\bot\ a\ 1)\oplus(\bot\ a\ 1))\otimes\boxed{{\scriptstyle\mathsf{mc}\uparrow}\dfrac{1\ a\ \bot}{(1\ a\ \bot)\ \&\ (1\ a\ \bot)}}}{\boxed{{\scriptstyle a\uparrow}\dfrac{(\bot\ a\ 1)\otimes(1\ a\ \bot)}{(\bot\otimes 1)\ a\ (1\otimes\bot)}}\oplus\boxed{{\scriptstyle a\uparrow}\dfrac{(\bot\ a\ 1)\otimes(1\ a\ \bot)}{(\bot\otimes 1)\ a\ (1\otimes\bot)}}}}{(\bot\otimes 1)\ a\ (1\otimes\bot)} \ .$$

or, written in terms of nestings:

$${\scriptstyle a\uparrow}\dfrac{\boxed{{\scriptstyle ac}\dfrac{(\bot\ a\ 1)\oplus(\bot\ a\ 1)}{\boxed{\dfrac{\bot\oplus\bot}{\bot}}\ a\ \boxed{\dfrac{1\oplus 1}{1}}}}\otimes(1\ a\ \bot)}{(\bot\otimes 1)\ a\ (1\otimes\bot)} \longrightarrow {\scriptstyle ac}\dfrac{{\scriptstyle\oplus\uparrow}\dfrac{((\bot\ a\ 1)\oplus(\bot\ a\ 1))\otimes\boxed{{\scriptstyle a\bar{c}}\dfrac{\boxed{\dfrac{1}{1\ \&\ 1}}\ a\ \boxed{\dfrac{\bot}{\bot\ \&\ \bot}}}{(1\ a\ \bot)\ \&\ (1\ a\ \bot)}}}{\boxed{{\scriptstyle a\uparrow}\dfrac{(\bot\ a\ 1)\otimes(1\ a\ \bot)}{(\bot\otimes 1)\ a\ (1\otimes\bot)}}\oplus\boxed{{\scriptstyle a\uparrow}\dfrac{(\bot\ a\ 1)\otimes(1\ a\ \bot)}{(\bot\otimes 1)\ a\ (1\otimes\bot)}}}}{\boxed{\dfrac{(\bot\otimes 1)\oplus(\bot\otimes 1)}{\boxed{\dfrac{\bot\oplus\bot}{\bot}}\otimes\boxed{\dfrac{1\oplus 1}{1}}}}\ a\ \boxed{\dfrac{(1\otimes\bot)\oplus(1\otimes\bot)}{\boxed{\dfrac{1\oplus 1}{1}}\otimes\boxed{\dfrac{\bot\oplus\bot}{\bot}}}}} \ .$$

Example 3.44. In SAMALLS we have the following reduction rule:

$$\otimes\!\uparrow\frac{\boxed{\otimes c\,\frac{((A\otimes B)\oplus(C\otimes D))}{((A\oplus C)\otimes(B\oplus D))}}\otimes(E\otimes F)}{((A\oplus C)\otimes E)\otimes((B\oplus D)\otimes F)} \longrightarrow \mathsf{mc}\!\downarrow\frac{\oplus\!\uparrow\frac{((A\otimes B)\oplus(C\otimes D))\otimes\boxed{\mathsf{mc}\!\uparrow\frac{E\otimes F}{(E\otimes F)\,\&\,(E\otimes F)}}}{\boxed{\otimes\!\uparrow\frac{(A\otimes B)\otimes(E\otimes F)}{(A\otimes E)\otimes(B\otimes F)}}\oplus\boxed{\otimes\!\uparrow\frac{(C\otimes D)\otimes(E\otimes F)}{(C\otimes E)\otimes(D\otimes F)}}}}{((A\oplus C)\otimes E)\otimes((B\oplus D)\otimes F)}$$

or, written in terms of nestings:

$$\otimes\!\uparrow\frac{\boxed{\otimes c\,\frac{((A\otimes B)\oplus(C\otimes D))}{((A\oplus C)\otimes(B\oplus D))}}\otimes(E\otimes F)}{((A\oplus C)\otimes E)\otimes((B\oplus D)\otimes F)} \longrightarrow \otimes c\,\frac{\oplus\!\uparrow\frac{((A\otimes B)\oplus(C\otimes D))\otimes\boxed{\otimes\bar{c}\,\frac{\boxed{\begin{array}{c}E\\ \phi_1\,\Vert\\ E\,\&\,E\end{array}}\otimes\boxed{\begin{array}{c}F\\ \phi_2\,\Vert\\ F\,\&\,F\end{array}}}{(E\otimes F)\,\&\,(E\otimes F)}}}{\boxed{\otimes\!\uparrow\frac{(A\otimes B)\otimes(E\otimes F)}{(A\otimes E)\otimes(B\otimes F)}}\oplus\boxed{\otimes\!\uparrow\frac{(C\otimes D)\otimes(E\otimes F)}{(C\otimes E)\otimes(D\otimes F)}}}}{\boxed{\otimes c\,\frac{(A\otimes E)\oplus(C\otimes E)}{(A\oplus C)\otimes\boxed{\begin{array}{c}E\oplus E\\ \psi_1\,\Vert\\ E\end{array}}}}\otimes\boxed{\otimes c\,\frac{(B\otimes F)\oplus(D\otimes F)}{(B\oplus D)\otimes\boxed{\begin{array}{c}F\oplus F\\ \psi_2\,\Vert\\ F\end{array}}}}}$$

where ϕ_1 and ϕ_2 are nestings of cocontractions and ψ_1 and ψ_2 are nestings of contractions.

Thus, we can easily see if a contraction permutes through another rule just by checking the existence of certain derivations, reducing the case by case analysis greatly. For example, we can see that in SAKS it is possible to move generic contractions with main relations $\wedge, a$ through every other possible rule. In SAMALLS it is possible to permute generic contractions with main relations $\otimes, a, \oplus$ through every rule.

Last, we can define the trivial reduction rule

$$i:\ \rho\,\frac{H\left\{\mathsf{mc}\!\downarrow\frac{A\ \nu\ B}{C}\right\}}{H'\,\{C\}} \quad\longrightarrow\quad \rho\,\frac{H\{A\ \nu\ B\}}{H'\left\{\mathsf{mc}\!\downarrow\frac{A\ \nu\ B}{C}\right\}}$$

We can easily extend the rewriting rules t_ρ presented to the symmetrical case

$$\rho\,\frac{(E\ \alpha'\ F)\ \beta\ \boxed{\mathsf{mc}\!\downarrow\frac{A\ \nu\ B}{C\ \alpha\ D}}}{(E\ \beta\ C)\ \alpha\ (F\ \beta'\ D)}\quad .$$

We take the duals of these reductions to present a reduction rule system to permute generic cocontractions upwards in a derivation.

For now we are not concerned about the general preservation of interpretability: we only want to permute merge contractions that correspond to merge contractions in the 'original' system. For example, for system SKS we simply want to present reduction rules to permute generic contractions composed of medials m, associativity and commutativity of $\vee$ and of atomic contractions $ac\downarrow$. We can take the representations of these generic contractions in SAKS and study the specific reductions for them. It is easy to see that these reductions are all interpretable.

In fact, if ρ does not involve atoms, then the contractions with main relation a remain untouched and are therefore still interpretable. Thus, we only need to study those reductions where contractions with main relation a appear, which is easily done: see examples 3.39, 3.41 and 3.43, which are all interpretable. Therefore, we can permute all generic contractions in SKS, and likewise in SMALLS. We will in fact use exactly these reductions in the next chapter to provide a procedure for cycle-elimination.

In this way, we can recover the rewriting systems C and Q of the previous section.

Definition 3.45. We define rewriting system C' for SAKS as the system given by the instances of the general reductions s, t, i for merge contractions of the form

$$\mathsf{mc}\!\downarrow \frac{(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{f}\ a\ \mathsf{t})}{\mathsf{f}\ a\ \mathsf{t}} \quad \text{and} \quad \mathsf{mc}\!\downarrow \frac{(\mathsf{t}\ a\ \mathsf{f}) \vee (\mathsf{t}\ a\ \mathsf{f})}{\mathsf{t}\ a\ \mathsf{f}} \quad .$$

We define rewriting system Q' for $\mathsf{SAMALLS}$ as the system given by the instances of the general reductions s, t, e for merge contractions γ of the form

$$\mathsf{mc}\!\downarrow \frac{(\perp\ a\ 1) \oplus (\perp\ a\ 1)}{\perp\ a\ 1} \quad \text{and} \quad \mathsf{mc}\!\downarrow \frac{(1\ a\ \perp) \oplus (1\ a\ \perp)}{1\ a\ \perp} \quad .$$

These systems correspond exactly to the rewriting systems defined in the previous section, and therefore termination can be proved in the same way. We define ai-cycles for SAKS and $\mathsf{SAMALLS}$ in identical fashion as in the previous section: they correspond to the connexion of an atomic introduction and an atomic cut.

Theorem 3.46. *Rewriting system C' is terminating on the set of ai-cycle-free derivations.*

Theorem 3.47. *Rewriting system Q' is terminating on the set of ai-cycle-free derivations.*

Furthermore, with these rules we can consider rewriting systems for SAKS and for $\mathsf{SAMALLS}$ that would allow us to obtain full decompositon theorems for classical logic and for multiplicative additive linear logic.

As we showed in Section 3.1, in SAKS and $\mathsf{SAMALLS}$ there are derivations with *ai*-cycles where the reductions for atomic contractions do not terminate. When considering the reduction rules for other relations, we increase the type of cycles that can lead to non-termination. However, in both SAKS and $\mathsf{SAMALLS}$ every such cycle will originate from the presence of a "critical medial" which we will define in the next chapter. By permuting the widest merge (co)contraction first we can therefore guarantee that it is not in a cycle, and thus we obtain a normalisation strategy. To prove termination we only need to find an adequate notion of *rank* for merge (co)contractions. Finding the appropriate notion of rank will be the focus of future research.

Definition 3.48. We define rewriting system D for SAKS as the system given by the general reductions s, t, i, the symmetric reductions, and the dual reductions for merge contractions with main relations $\wedge, \vee, a$.

Conjecture 3.49. *System* D *is weakly normalising on tame proofs.*

Normalisation for $\mathsf{SAMALLS}$ is slightly more complex: generic contractions with main relation $\parr$ do not permute with the associativity rule for $\parr$ as the rule $\dfrac{(A \oplus B) \parr (C \mathbin{\&} D)}{(A \parr C) \oplus (B \parr D)}$, required for permutation in Definition 3.42 is not in the system. Thus, the focus of the reduction should be to permute every other generic contraction.

Definition 3.50. We define rewriting system G for $\mathsf{SAMALLS}$ as the system given by the general reductions s, t, i, the symmetric reductions, and the dual reductions for generic contractions with main relations $\otimes, \&, a, \oplus$.

Conjecture 3.51. *System* G *is weakly normalising on tame proofs.*

In both systems the decomposition results affecting atomic (co)weakenings are very simple, since every reduction rule reduces the number of atoms in a derivation. Therefore, once the reductions of D and G have been applied, atomic weakenings can be permuted since they do not introduce any new generic (co)contractions as we noted in the previous section. Unitary weakenings remain in the proof, but they can in most cases be replaced by instances of linear rules: in classical logic for example, the inference $\dfrac{\mathsf{f}}{\mathsf{t}}$ can be obtained from the rule $\wedge\!\downarrow$.

3.3 Conclusions

By presenting these general reduction rules we have shown that the atomic decomposition results for classical logic and linear logic are in fact a particular case of a wider phenomenon: both rewriting systems exploit the shape of atomic contractions to be able to permute them with other rules.

Furthermore, by being able to permute generic contractions together, we advance towards proving a full decomposition theorem for classical logic and multiplicative additive linear logic, which will be the focus of future research.

Another area of further research will be the exploration of the similarities between the general reduction rules that we presented and the duplication rules for sharing graphs [22]. In fact these similarities are perhaps not so surprising, since there is a Curry–Howard correspondence between well-formed interaction nets and a deep-inference deduction system based on linear logic [17]: decomposition in this system via the general rules of this chapter might well correspond to the duplication rules of sharing graphs.

In the next chapter we will present an application of the general reduction rules: the elimination of *ai*-cycles in both logics as a local procedure.

Chapter 4

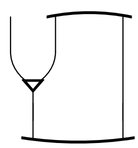
Removing cycles

As we saw in the previous chapter, atomic contractions and atomic cocontractions can be permuted downwards/upwards in a classical logic derivation in the absence of ai-cycles. Identically, the result holds for multiplicative additive linear logic.

Our goal in this chapter is to take advantage of the reductions presented in the previous chapter to show that we can remove ai-cycles without recurring to cut-elimination, therefore proving the independence of the decomposition and the cut-elimination procedures.

Furthermore, the phenomenon of cycles has been studied in the sequent calculus, where it has been shown that it is possible to remove them through a procedure of quadratic-time complexity [12]. With the procedure we present in what follows, we hope to be able to study the complexity cost of cycle-elimination in deep inference in future research.

Cycles are a particular construction caused by the 'connection' of an introduction and a cut, as we saw in Section 3.1:



In the sequent calculus, cycles can only occur due to the presence of contractions [11]. Likewise, in our case they exist due to the presence of the medial rule m, fundamental to make contraction atomic.

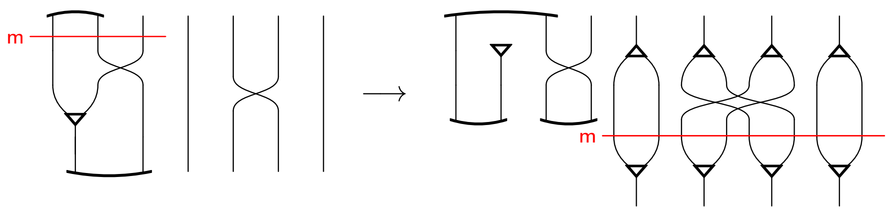
The intuition behind this procedure is simple. For an ai-cycle to occur in classical logic, two edges of an atomic flow that were related by $\vee$ at the top of the flow have to be connected by $\wedge$ at the bottom of the flow. Therefore, an instance of a rule that changes the relation between formulae from $\vee$ to $\wedge$ needs to occur, and it must contain the atoms involved in the cycle. In SKS, the only such rule is m.

$$\mathsf{m}\,\frac{(A\{a\}\wedge B)\vee(C\wedge D\{\bar{a}\})}{(A\{a\}\vee C)\wedge(B\vee D\{\bar{a}\})}$$

Likewise, for an ai-cycle to occur in multiplicative additive linear logic, an instance of a rule that changes the main relation between formulae from $\alpha \neq \otimes$ to $\otimes$ has to occur. The only such rule is $\otimes c$.

$$\otimes c\,\frac{(A\{a\}\otimes B)\oplus(C\otimes D\{\bar{a}\})}{(A\{a\}\oplus C)\otimes(B\oplus D\{\bar{a}\})}$$

Following this observation, and with the reduction rules of the previous section as tools, the procedure to remove cycles is very simple. We can easily permute these critical instances of generic contractions with main connective $\wedge$ or $\otimes$ downwards in a proof until they are below the cut of the cycle. When at the end of the procedure there are no remaining critical contractions above the cut, the cycle will have disappeared. The permutations can however introduce significant changes in a derivation beyond simply removing cycles: this is illustrated in the flow below by an example of how the atomic flow of a derivation can be modified by the cycle-removal procedure.



This idea of removing cycles by starting from the 'critical medial' has in fact yielded two methods for the elimination of cycles: the one presented in what follows, and the one presented in [1], that will both be studied to ascertain the complexity cost of each procedure.

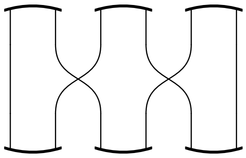
We present the cycle-elimination procedure for SKS, but it can be straightforwardly adapted to SLLS.

4.1 Unicycles

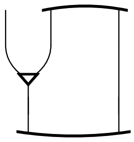
For a critical medial to modify the logical relation between the atom occurrences involved in the cycle, we need to be in the case of a cycle with a single atomic identity $ai{\downarrow}$ and a single cut $ai{\uparrow}$, as opposed to a cycle involving atoms coming from several identities.

Definition 4.1. We say that an ai-cycle is a *unicycle* when it only crosses a single $ai\!\downarrow$ node and a single $ai\!\uparrow$ node. We say that an ai-cycle is an *n-multicycle* when it crosses n distinct $ai\!\downarrow$ nodes and n distinct $ai\!\uparrow$ nodes.

Example 4.2. The cycle below is a 3-multicycle.



The cycle below is a unicycle.



Whereas in unicycles it is clear that the the logical relation between the two atom occurrences involved in the cycle must change from a conjunction $\vee$ to a disjunction $\wedge$, in multicycles it is not necessarily so. It is however easy to transform multicycles into unicycles by standard transformations often used to manipulate SKS derivations, as in the proof of the following Lemma 4.3 that can be found in [32].

Lemma 4.3. *For any formula context $H\{\ \}$ and any formula A there is a derivation*

$$\begin{array}{c} H\{\mathsf{t}\} \wedge A \\ \phi \,\Vert\, \{=,s\} \\ H\{A\} \end{array}$$

of size quadratic on the size of $H\{A\}$.

Lemma 4.4. *Given a derivation ϕ with an n-multicycle, there exists a derivation ψ of size cubic on the size of ϕ where the multicycle is replaced by n unicycles.*

Proof. Let $\begin{array}{c} A \\ \phi \,\Vert \\ B \end{array}$ be a derivation with an n-multicycle. We denote by a_1, $\bar{a}_1$, …, a_n, $\bar{a}_n$ the occurrences of atoms involved in the n-multicycle.

For each identity in the n-multicycle we perform the reduction

$$\begin{array}{c} A \\ \phi_i \,\Vert \\ K\left\{ \boxed{\mathsf{ai}\!\downarrow \dfrac{\mathsf{t}}{a_i \vee \bar{a}_i}} \right\} \end{array} \longrightarrow \begin{array}{c} = \dfrac{A}{\boxed{\begin{array}{c} A \\ \phi_i \,\Vert \\ K\{\mathsf{t}\} \end{array}} \wedge \boxed{\mathsf{ai}\!\downarrow \dfrac{\mathsf{t}}{a_i \vee \bar{a}_i}}} \\ \phi_i' \,\Vert \\ K\{a_i \vee \bar{a}_i\} \end{array} ,$$

where ϕ'_i is obtained as per Lemma 4.3, to obtain a derivation of the shape

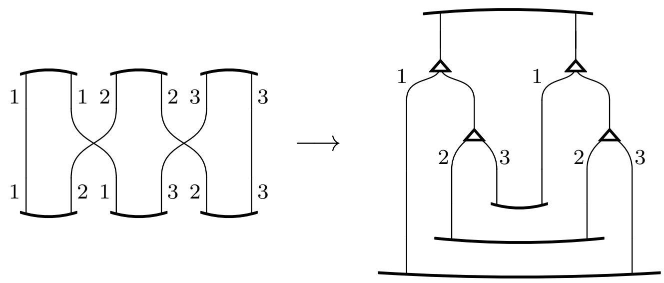
$$\begin{array}{c}\boxed{A \wedge \boxed{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{a_1 \vee \bar{a}_1}} \wedge \boxed{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{\bar{a}_2 \vee a_2}} \wedge \cdots \wedge \boxed{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{t}}{\bar{a}_n \vee a_n}}} \\ \phi' \, \| \\ B \end{array} .$$

Then, we take ψ to be:

$$\psi \equiv \begin{array}{c}\boxed{A \wedge \boxed{\mathsf{m}\,\dfrac{\boxed{\dfrac{\mathsf{t}}{\boxed{\mathsf{ac}{\uparrow}\,\dfrac{a}{a_1 \wedge a}} \vee \boxed{\mathsf{ac}{\uparrow}\,\dfrac{\bar{a}}{\bar{a}_1 \wedge \bar{a}}}}}}{(a_1 \vee \bar{a}_1) \wedge \boxed{\begin{array}{c}\boxed{\mathsf{ac}{\uparrow}\,\dfrac{a}{a_2 \wedge a}} \vee \boxed{\mathsf{ac}{\uparrow}\,\dfrac{\bar{a}}{\bar{a}_2 \wedge \bar{a}}} \\ \| \{\mathsf{ac}{\uparrow}, \mathsf{m}\} \\ (a_2 \vee \bar{a}_2) \wedge \cdots \wedge (a_n \vee \bar{a}_n)\end{array}}}}} \\ \phi' \, \| \\ B \end{array} .$$

□

In this way, we transform an n-multicycle into n unicycles that all share an identity. For example:



Definition 4.5. Let ϕ be a derivation containing a unicycle, represented by the ai-cycle $\epsilon_1, \dots, \epsilon_n$ in its atomic flow. The *critical medial* for this unicycle is the lowest instance of a rule

$$\mathsf{m}\,\frac{(A\{a\} \wedge B) \vee (C \wedge D\{\bar{a}\})}{(A\{a\} \vee C) \wedge (B \vee D\{\bar{a}\})}$$

in ϕ where the occurrences of a and $\bar{a}$ are represented in the atomic flow by one of the edges belonging to the ai-cycle $\epsilon_1, \dots, \epsilon_n$.

$$
\mathsf{m}\frac{(A\{a\} \wedge B) \vee (C \wedge D\{\bar{a}\})}{(A\{a\} \vee C) \wedge (B \vee D\{\bar{a}\})}
$$

A simple study of SKS rules shows that the medial rule m is the only rule that can change the connective between formulae from a disjunction to a conjunction. Since in unicycles the logical relation between two atom occurrences involved in the cycle must change from a conjunction $\vee$ to a disjunction $\wedge$ before they are connected in the cut rule $\mathsf{ai}\uparrow$, every unicycle contains a critical medial.

4.2 Cycle removal procedure

We choose to implement the procedure to remove cycles atomically rather than subatomically for ease of following the flows. It is very straightforward to adapt the rewriting rules presented in the previous chapter to system SKS.

Definition 4.6. We define the rules

$$
\begin{array}{cc}
\vee\!\downarrow\frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)} & \wedge\!\uparrow\frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)} \\
\wedge\!\downarrow\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} & \vee\!\uparrow\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}
\end{array} .
$$

Proposition 4.7. *The rules of Definition 4.6 are admissible in* SKS.

Proof. Admissibility of $\vee\!\downarrow$ and $\wedge\!\uparrow$ is clear by associativity and commutativity of $\vee$ and $\wedge$. The rules $\vee\!\uparrow$ and $\wedge\!\downarrow$ are derivable from two applications of the rule s. □

Proposition 4.8. *In a* SKS *derivation, we can replace every instance of associativity and commutativity of* $\vee$ *by instances of the rule* $\vee\!\downarrow$ *and the unit rule for* $\vee$*. Likewise we can replace every instance of associativity and commutativity of* $\wedge$ *by instances of the rule* $\wedge\!\uparrow$ *and the unit rule for* $\wedge$*.*

Furthermore, in a SKS *derivation we can replace every instance of the rule* s *by instances of the rule* $\wedge\!\downarrow$ *and the unit rule for* $\vee$*, or by instances of the rule* $\vee\!\uparrow$ *and the unit rule for* $\wedge$*.*

We will consider system SKS with the rule s and the associativity and commutativity rules for $\vee$ and $\wedge$ replaced by the rules $\wedge\!\downarrow$, $\vee\!\uparrow$, $\vee\!\downarrow$ and $\wedge\!\uparrow$. This small change does not affect significantly the size of derivations, and will not warrant a change of name – we will still refer to these derivations as SKS derivations.

We easily replicate the definitions of Section 4 for non-subatomic systems.

For example:

Definition 4.9. An $\vee$-*merge* of two formulae is defined as follows:

- $A \vee B$ is an $\vee$-merge of A and B that we call a *trivial merge*;
- a is an $\vee$-merge of a and a, where a is an atom or a unit t or f;
- $C_1 \ \alpha \ C_2$ is an $\vee$-merge of $A_1 \ \alpha \ A_2$ and $B_1 \ \alpha \ B_2$ for $\alpha \in \{\vee, \wedge\}$ if C_1 is an $\vee$-merge of A_1 and B_1 and C_2 is an $\vee$-merge of A_2 and B_2.

If C is an $\vee$-merge of A and B, by an abuse of language we will sometimes refer to the triple (A, B, C) as an $\vee$-merge.

$\wedge$-*merges* of two formulae are defined as follows:

- $A \wedge B$ is an $\wedge$-merge of A and B that we call a *trivial merge*;
- a is an $\wedge$-merge of a and a, where a is an atom or a unit t or f;
- $C_1 \ \alpha \ C_2$ is an $\wedge$-merge of $A_1 \ \alpha \ A_2$ and $B_1 \ \alpha \ B_2$ for $\alpha \in \{\vee, \wedge\}$ if C_1 is an $\wedge$-merge of A_1 and B_1 and C_2 is an $\wedge$-merge of A_2 and B_2.

Definition 4.10. $\mathsf{mc}\!\downarrow \dfrac{A \vee B}{C}$ is a *merge contraction* if C is a non-trivial $\vee$-merge of A and B.

$\mathsf{mc}\!\uparrow \dfrac{C}{A \wedge B}$ is a *merge cocontraction* if C is a non-trivial $\wedge$-merge of A and B.

Clearly, merge contractions correspond to nestings of the rules m, $ac\!\downarrow$ and $\vee\!\downarrow$. We can identify these nestings in a proof and simply replace them by merge contractions. Likewise, we can replace merge contractions by nestings. We will abuse language and interchangeably use the word nesting or merge contraction for this type of structure.

The following proposition can be proved by an obvious structural induction on the formula.

Proposition 4.11. *For any formula A there is a merge contraction*

$$\mathsf{mc}\!\downarrow \frac{A \vee A}{A} \quad .$$

Dually there is a merge cocontraction

$$\mathsf{mc}\!\uparrow \frac{A}{A \wedge A} \quad .$$

Projections can be defined identically to the subatomic case.

Definition 4.12. If C is a ν-merge of A and B, we define the *projections* $\begin{array}{c} A \\ \pi_A \, \| \, \{=, aw\!\downarrow\} \\ C \end{array}$ *and* $\begin{array}{c} B \\ \pi_B \, \| \, \{=, aw\!\downarrow\} \\ C \end{array}$ *associated to the merge* recursively as follows:

- if $C \equiv A \vee B$, we take $\pi_A \equiv {=}\dfrac{A}{A \vee \boxed{\begin{array}{l} \mathsf{f} \\ \Vert \{=, aw\downarrow\} \\ B \end{array}}}$ and $\pi_B \equiv {=}\dfrac{B}{\boxed{\begin{array}{l} \mathsf{f} \\ \Vert \{=, aw\downarrow\} \\ A \end{array}} \vee B}$;

- if $C \equiv A \equiv B \equiv a$ with a a unit or an atom, we take $\pi_A \equiv a$ and $\pi_B \equiv a$;

- if $C \equiv C_1 \ \alpha\ C_2$, $A \equiv A_1 \ \alpha\ A_2$, $B \equiv B_1 \ \alpha\ B_2$ where $\alpha \in \{\vee, \wedge\}$ and C_1 is an $\vee$-merge of A_1 and B_1 and C_2 is an $\vee$-merge of A_2 and B_2, we take

$$\pi_A \equiv \boxed{\begin{array}{c} A_1 \\ \pi_{A_1} \Vert \{=, aw\downarrow\} \\ C_1 \end{array}} \ \alpha\ \boxed{\begin{array}{c} A_2 \\ \pi_{A_2} \Vert \{=, aw\downarrow\} \\ C_2 \end{array}} \qquad \text{and} \qquad \pi_B \equiv \boxed{\begin{array}{c} B_1 \\ \pi_{B_1} \Vert \{=, aw\downarrow\} \\ C_1 \end{array}} \ \alpha\ \boxed{\begin{array}{c} B_2 \\ \pi_{B_2} \Vert \{=, aw\downarrow\} \\ C_2 \end{array}} \quad ,$$

  where π_{A_1}, π_{B_1} are the projections associated to the merge (A_1, B_1, C_1) and π_{A_2}, π_{B_2} are the projections associated to the merge (A_2, B_2, C_2).

Since in unicycles the logical relation between two atom occurrences involved in the cycle must change from a conjunction $\vee$ to a disjunction $\wedge$ before they are connected in the cut rule $ai\uparrow$, every unicycle contains a critical merge contraction where this relation is changed.

Definition 4.13. Let ϕ be a derivation. A *critical merge contraction* is a maximal merge contraction that contains a critical medial.

We will permute critical merge contractions downwards in a proof, until they are no longer in a cycle. We will do so with the reduction rules s and t defined in the previous chapter applied to SKS. However, since we are only permuting critical merge contractions and leaving all the other contraction rules as they were, we may need to permute the critical merge contraction through instances of m, $\vee\downarrow$ or $ac\downarrow$.

Definition 4.14. We define system CR for SKS as the rewriting system given by the following reduction rules:

$$t: \quad \rho\,\dfrac{\boxed{\mathsf{mc}\!\downarrow \dfrac{(A_1\ \alpha\ A_2) \vee (B_1\ \alpha\ B_2)}{C_1\ \alpha\ C_2}} \wedge (D\ \beta\ E)}{(C_1 \wedge D)\ \delta\ (C_2\ \epsilon\ E)} \quad \longrightarrow \quad \mathsf{mc}\!\downarrow \dfrac{\vee\!\uparrow \dfrac{((A_1\ \alpha\ A_2) \vee (B_1\ \alpha\ B_2)) \wedge \boxed{\mathsf{mc}\!\uparrow \dfrac{(D\ \beta\ E)}{(D\ \beta\ E) \wedge (D\ \beta\ E)}}}{\boxed{\rho\,\dfrac{(A_1\ \alpha\ A_2) \wedge (D\ \beta\ E)}{(A_1 \wedge D)\ \delta\ (A_2\ \epsilon\ E)}} \vee \boxed{\rho\,\dfrac{(B_1\ \alpha\ B_2) \wedge (D\ \beta\ E)}{(B_1 \wedge D)\ \delta\ (B_2\ \epsilon\ E)}}}}{(C_1 \wedge D)\ \delta\ (C_2\ \epsilon\ E)}$$

where ρ is an instance of $\wedge\uparrow$, $\wedge\downarrow$ or $\vee\uparrow$.

$$r:\quad \rho\,\dfrac{\boxed{\mathsf{mc}{\downarrow}\,\dfrac{(A_1 \;\alpha\; A_2) \vee (B_1 \;\alpha\; B_2)}{C_1 \;\alpha\; C_2}} \vee (D \;\beta\; E)}{(C_1 \vee D) \;\delta\; (C_2 \vee E)} \quad\longrightarrow\quad \mathsf{mc}{\downarrow}\,\dfrac{{\vee}{\downarrow}\,\dfrac{((A_1 \;\alpha\; A_2) \vee (B_1 \;\alpha\; B_2)) \vee \left((D \;\beta\; E) \vee \boxed{\begin{array}{c}\mathsf{f}\\ \Vert\{=,\mathsf{aw}{\downarrow}\}\\ (D \;\beta\; E)\end{array}}\right)}{\boxed{\rho\,\dfrac{(A_1 \;\alpha\; A_2) \vee (D \;\beta\; E)}{((A_1 \vee D))\;\delta\;(A_2 \vee E)}} \vee \boxed{\rho\,\dfrac{(B_1 \;\alpha\; B_2) \vee (D \;\beta\; E)}{(B_1 \vee D)\;\delta\;(B_2 \vee E)}}}}{(C_1 \vee D) \;\delta\; (C_2 \vee E)}$$

where ρ is an instance of m or ${\vee}{\downarrow}$.

$$s:\quad \mathsf{mc}{\downarrow}\,\dfrac{A \vee B}{C\left\{\rho\,\dfrac{O}{P}\right\}} \quad\longrightarrow\quad \mathsf{mc}{\downarrow}\,\dfrac{\boxed{\begin{array}{c}A\\ \pi_A \Vert\{=,\mathsf{aw}{\downarrow}\}\\ C\left\{\rho\,\dfrac{O}{P}\right\}\end{array}} \vee \boxed{\begin{array}{c}B\\ \pi_B \Vert\{=,\mathsf{aw}{\downarrow}\}\\ C\left\{\rho\,\dfrac{O}{P}\right\}\end{array}}}{C\{P\}}$$

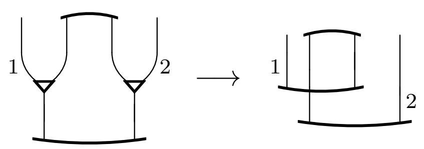
where ρ is any rule, and π_A and π_B are the projections associated to the merge $(A, B, C\{O\})$.

We define the trivial reduction:

$$i:\quad \rho\,\dfrac{H\left\{\mathsf{mc}{\downarrow}\,\dfrac{Q}{S}\right\}}{H'\{S\}} \quad\longrightarrow\quad \rho\,\dfrac{H\{Q\}}{H'\left\{\mathsf{mc}{\downarrow}\,\dfrac{Q}{S}\right\}}$$

To remove ai-cycles from a derivation, we will start by transforming multicycles into unicycles by applying Lemma 4.4. In this way, we ensure that every cycle has a critical merge contraction, that we will permute downwards until it disappears and breaks the cycle, by applying the reductions of sytem CR.

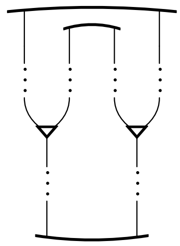
To ensure termination, we need to make sure that in this process we break cycles and do not create any new ones. Cycles will be broken when the critical merge contraction is permuted below the cut belonging to its cycle in an application of rule s exactly as presented in example 3.41. We will transform a single cut into two cuts.



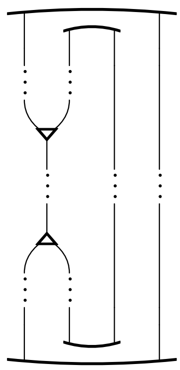
To ensure that this rewriting does indeed break the cycle and doesn't simply create a new one, we need to make sure that edges 1 and 2 are not connected by an atomic identity. As can be seen quite intuitively from the flow, that is the case only when in the original derivation there exist two cycles that share a cut. However, we can easily transform derivations in such a way that cycles never share a cut.

Lemma 4.15. *Given a derivation ϕ with two unicycles with different identities that share a cut, there exists a derivation ψ with the same premiss and conclusion where they are replaced by two unicycles that do not share a cut.*

Proof. The two unicycles that share a cut must be connected by a contraction on each edge.



We move one of these contractions downwards by applying the reductions of rewriting system C until it goes through the cut.



□

In a proof without multicycles and where no two cycles share a cut, termination of the cycle-elimination procedure will be guaranteed. We will show termination of the procedure by proving we can remove critical merge contractions one by one. Since termination can be quite intuitively understood from the changes induced on the flow of the derivation by the cycle-elimination procedure, we will accompany the proof of the following Lemma 4.16 with a study of the changes on flows that each rewriting rule of system M produces. An accurate formal bound for the cost of the procedure has yet to be established, but the study of the flow changes is expected to provide us with the necessary intuition to obtain it.

Theorem 4.16. *Let ϕ be a derivation with no multicycles and where no two cycles with different identities share a cut. If ϕ contains n critical merge contractions, there exists a derivation ψ with the same premiss and conclusion with $n-1$ critical merge contractions. Furthermore, ψ contains no multicycles and no two cycles with different identities share a cut.*

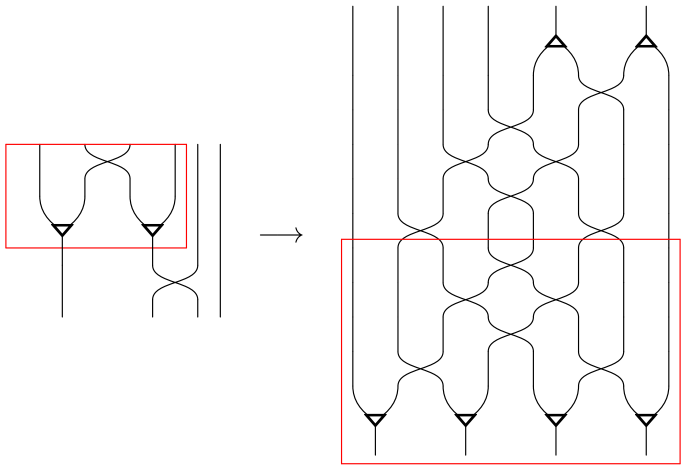
Proof. We consider the lowest critical merge contraction of ϕ, that we call M. We apply a reduction of system CR to permute M downwards in ϕ. We repeat this process until we obtain a derivation ψ where M is not a critical merge contraction as it has been permuted below the cuts of the cycles whose critical medials it contained.

At every application of a reduction of CR, the number of inference rules below M is decreased: the procedure therefore terminates, and at the end of it M will no longer be critical. We only need to show that we do not create multicycles, cycles with different identities that share a cut, or new critical merge contractions.

We will show that through a study of the flows. We will enclose the parts of the flow that belong to the critical merge contraction in a *red* box.

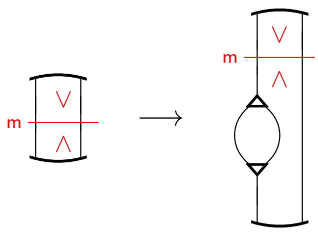
We call ρ the rule instance below the critical merge contraction.

- Instances of t do not change the links between the existing edges of a flow. They may bifurcate previously "single" edges.



  If the edges that are bifurcated do not belong to a cycle, the number of cycles and their identities and cuts remain unchanged. However, if the edge of a cycle is bifurcated we create new cycles. We do not, however, create new critical merge contractions:

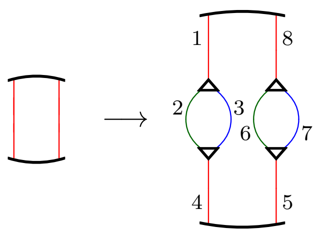
  - If only one edge of a cycle is bifurcated, since the critical merge contraction for the cycle is above the bifurcation, it is the critical merge contraction for the new cycle as well.



– If two edges of the cycle are bifurcated, we are in the following situation:

$$\begin{array}{c}
P\left\{\dfrac{\mathsf{t}}{{\color{red}a}\vee{\color{red}\bar a}}\right\}\\
\|\\
S\left\{\rho\,\dfrac{\boxed{M\,\dfrac{((A_1\;\alpha\;A_2))\vee((B_1\;\alpha\;B_2))}{C_1\;\alpha\;C_2}}\wedge K\{{\color{red}a}\}\{{\color{red}\bar a}\}}{F\{{\color{red}a}\}\{{\color{red}\bar a}\}}\right\}\\
\|\\
T\left\{\dfrac{{\color{red}a}\wedge{\color{red}\bar a}}{\mathsf{f}}\right\}
\end{array}\quad\longrightarrow$$

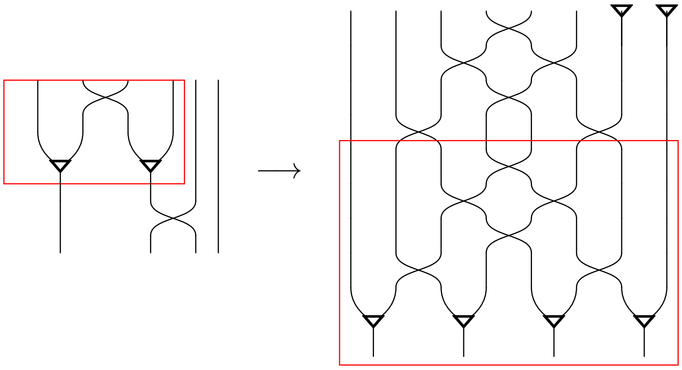
$$\begin{array}{c}
P\left\{\dfrac{\mathsf{t}}{{\color{red}a}\vee{\color{red}\bar a}}\right\}\\
\|\\
S\left\{M\,\dfrac{\mathsf{s}\!\uparrow\dfrac{(((A_1\;\alpha\;A_2)\vee(B_1\;\alpha\;B_2)))\wedge\boxed{\mathsf{mc}\!\uparrow\dfrac{K\{{\color{red}a}\}\{{\color{red}\bar a}\}}{K\{{\color{green}a}\}\{{\color{green}\bar a}\}\wedge K\{{\color{blue}a}\}\{{\color{blue}\bar a}\}}}}{\boxed{\rho\,\dfrac{((A_1\;\alpha\;A_2))\wedge K\{{\color{green}a}\}\{{\color{green}\bar a}\}}{F\{{\color{green}a}\}\{{\color{green}\bar a}\}}}\vee\boxed{\rho\,\dfrac{((B_1\;\alpha\;B_2))\wedge K\{{\color{blue}a}\}\{{\color{blue}\bar a}\}}{F\{{\color{blue}a}\}\{{\color{blue}\bar a}\}}}}}{F\{{\color{red}a}\}\{{\color{red}\bar a}\}}\right\}\\
\|\\
T\left\{\dfrac{{\color{red}a}\wedge{\color{red}\bar a}}{\mathsf{f}}\right\}
\end{array}$$



In this case, the critical merge contraction for the original cycle (on the left) is now the critical merge contraction for the cycles $(1,2,4,5,6,8)$ and $(1,3,4,5,7,8)$ where a and $\bar a$ are of the same color. M is the critical merge contraction for the cycles $(1,2,4,5,7,8)$ and $(1,3,4,5,6,8)$ where a and $\bar a$ are of different colors. Thus, although we do add cycles, we do not add critical merge contractions.

Complexity is generated in the cycle-removing procedure by turning straight edges into ‘sausages’.

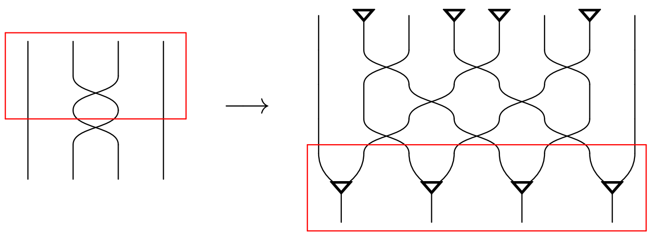
- Instances of r do not change the links between the existing edges of a flow. They introduce some contractions where one edge is connected to a weakening. Therefore the application of this rule cannot create new cycles (i.e. create new critical merge contractions) or change the identities or cuts of cycles.



  The size of the proof after the cycle elimination will not be increased significantly by the application of these reductions, since the weakenings can be pulled down, and the edges that have been connected to a weakening node will return to simply being straight edges.

- Instances of s where ρ is a rule that does not involve atoms do not change the links between the existing edges of a flow. They merely create two 'smaller' instances of ρ that do not involve atoms and do not break or change any existing connections. Therefore the application of these rules cannot create new cycles (i.e. create new critical merge contractions) or change the identities or cuts of cycles.

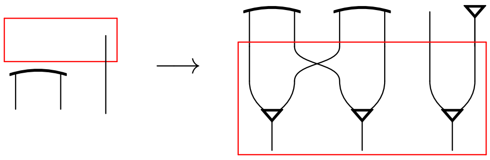
  They might introduce some contractions where one edge is connected to a weakening.



  Like in the above case for r, this does not introduce significant complexity since weakenings can be pulled down.

- Instances of s where $\rho = \mathsf{ai}{\downarrow}$ change a single introduction into two introductions and two contractions, and may introduce some contractions where one edge is connected to a weakening.

$$\mathsf{mc}{\downarrow}\frac{A \vee B}{C\left\{\mathsf{ai}{\downarrow}\dfrac{\mathsf{t}}{a\vee\bar a}\right\}} \longrightarrow \mathsf{mc}{\downarrow}\frac{\boxed{\begin{array}{c}A\\ \pi_A\,\Vert\,\{=,\mathsf{aw}{\downarrow}\}\\ C\left\{\mathsf{ai}{\downarrow}\dfrac{\mathsf{t}}{a\vee\bar a}\right\}\end{array}} \vee \boxed{\begin{array}{c}B\\ \pi_B\,\Vert\,\{=,\mathsf{aw}{\downarrow}\}\\ C\left\{\mathsf{ai}{\downarrow}\dfrac{\mathsf{t}}{a\vee\bar a}\right\}\end{array}}}{C\,\{a\vee\bar a\}}$$

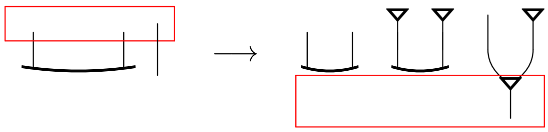


This reduction could only introduce a cycle if the instance of $\mathsf{ai}{\downarrow}$ being permuted was part of a cycle. This is however not a possible case since we are permuting the lowermost critical merge contraction, and if the instance of $\mathsf{ai}{\downarrow}$ was part of a cycle there would be a critical merge contraction that is lower.

- Instances of s where $\rho = \mathsf{ai}{\uparrow}$ duplicate cuts and remove atomic contractions. There are two possible cases:

  – The first option is given by the reduction below or its symmetrical case.

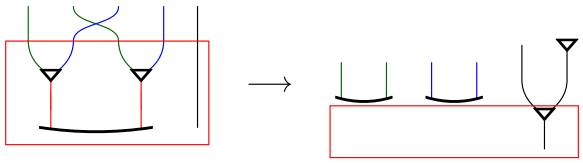
$$\mathsf{mc}{\downarrow}\frac{A\{a\wedge\bar a\}\vee B\{M\}}{C\left\{\mathsf{ai}{\uparrow}\dfrac{a\wedge\bar a}{\mathsf{f}}\vee M\right\}} \longrightarrow \mathsf{mc}{\downarrow}\frac{\boxed{\begin{array}{c}A\\ \Vert\,\{=,\mathsf{aw}{\downarrow}\}\\ C\left\{\boxed{\mathsf{ai}{\uparrow}\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee\boxed{\begin{array}{c}\mathsf{f}\\ \Vert\,\{=,\mathsf{aw}{\downarrow}\}\\ M\end{array}}\right\}\end{array}} \vee \boxed{\begin{array}{c}B\\ \Vert\,\{=,\mathsf{aw}{\downarrow}\}\\ C\left\{\boxed{\mathsf{ai}{\uparrow}\dfrac{\boxed{\mathsf{aw}{\downarrow}\dfrac{\mathsf{f}}{a}}\wedge\boxed{\mathsf{aw}{\downarrow}\dfrac{\mathsf{f}}{\bar a}}}{\mathsf{f}}}\vee M\right\}\end{array}}}{C\,\{\mathsf{f}\vee M\}}$$



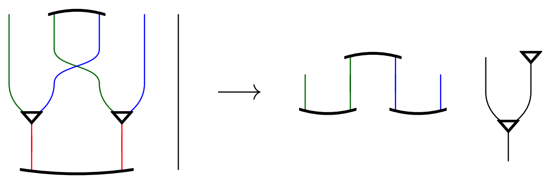
A cut whose edges are connected to weakenings is introduced, and some contractions where one edge is connected to a weakening may be introduced as well. Therefore the application of these rules cannot create new cycles (i.e. create new critical merge contractions) or change the identities or cuts of cycles.

– The second option is given by

$$\mathsf{M}\,\frac{A\{a \wedge \bar{a}\} \vee B\{a \wedge \bar{a}\}}{C\left\{\mathsf{ai}\!\uparrow\frac{a \wedge \bar{a}}{\mathsf{f}}\right\}} \quad\longrightarrow\quad \mathsf{M}\,\frac{\boxed{\begin{matrix} A \\ \pi_A \,\|\, \{=,\mathsf{aw}\!\downarrow\} \\ C\left\{\mathsf{ai}\!\uparrow\frac{a \wedge \bar{a}}{\mathsf{f}}\right\}\end{matrix}} \vee \boxed{\begin{matrix} B \\ \pi_B \,\|\, \{=,\mathsf{aw}\!\downarrow\} \\ C\left\{\mathsf{ai}\!\uparrow\frac{a \wedge \bar{a}}{\mathsf{f}}\right\}\end{matrix}}}{C\,\{\mathsf{f}\}}$$



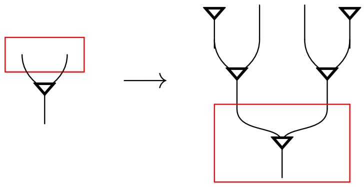
If a and $\bar{a}$ of the same color are in a cycle, then the critical merge contraction for the cycle is above the subderivation that we are rewriting, and remains the critical medial for the cycle. If a and $\bar{a}$ of different colors belong to a cycle, then M is its critical merge contraction, and the cycle is broken through this reduction: since there are no cycles with different identities sharing a cut, the other two edges cannot be connected.



Therefore the application of this rule reduces or maintains the number of critical merge contractions), and does not create multicycles or cycles with different identities that share a cut.

- Applications of s where $\rho = \mathsf{ac}\!\downarrow$ do not change ai-connexions. They may introduce some contractions where one edge is connected to a weakening. Therefore an application of this rule cannot create new cycles (i.e. create new critical merge contractions) or change the identities or cuts of cycles.

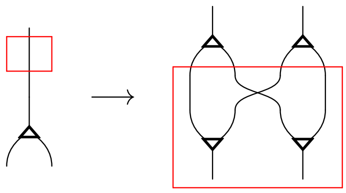
$$\mathsf{mc}\!\downarrow\,\frac{A \vee B}{C\left\{ac\!\downarrow\frac{a \vee a}{a}\right\}} \quad\longrightarrow\quad \mathsf{mc}\!\downarrow\,\frac{\boxed{\begin{matrix} A \\ \pi_A \,\|\, \{=,aw\!\downarrow\} \\ C\left\{ac\!\downarrow\frac{a \vee a}{a}\right\}\end{matrix}} \vee \boxed{\begin{matrix} B \\ \pi_B \,\|\, \{=,aw\!\downarrow\} \\ C\left\{ac\!\downarrow\frac{a \vee a}{a}\right\}\end{matrix}}}{C\,\{a\}}$$



Like in the above case for r, this does not introduce significant complexity since weakenings can be pulled down.

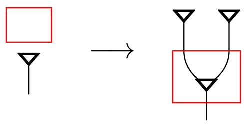
- Instances of s where $\rho = \mathsf{ac}\!\uparrow$ do not change ai-connexions. They may introduce some contractions and cocontractions where one edge is connected to a weakening.Therefore an application of this rule cannot create new cycles (i.e. create new critical merge contractions) or change the identities or cuts of cycles.

$$\mathsf{mc}\!\downarrow \frac{A \vee B}{C\left\{ ac\!\uparrow \dfrac{a}{a \wedge a} \right\}} \longrightarrow \mathsf{mc}\!\downarrow \dfrac{\boxed{\begin{array}{c} A \\ \pi_A \, \| \, \{=, aw\!\downarrow\} \\ C\left\{ ac\!\uparrow \dfrac{a}{a \wedge a} \right\} \end{array}} \vee \boxed{\begin{array}{c} B \\ \pi_B \, \| \, \{=, aw\!\downarrow\} \\ C\left\{ ac\!\uparrow \dfrac{a}{a \wedge a} \right\} \end{array}}}{C\{a \wedge a\}}$$



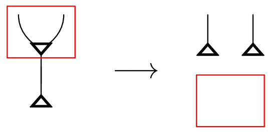
- Instances of s where $\rho = \mathsf{aw}\!\downarrow$ create a contraction topped by weakenings. They cannot produce new cycles.

$$\mathsf{mc}\!\downarrow \frac{A \vee B}{C\left\{ \mathsf{aw}\!\downarrow \dfrac{\mathsf{f}}{a} \right\}} \longrightarrow \mathsf{mc}\!\downarrow \dfrac{\boxed{\begin{array}{c} A \\ \pi_A \, \| \, \{=, \mathsf{aw}\!\downarrow\} \\ C\left\{ \mathsf{aw}\!\downarrow \dfrac{\mathsf{f}}{a} \right\} \end{array}} \vee \boxed{\begin{array}{c} B \\ \pi_B \, \| \, \{=, \mathsf{aw}\!\downarrow\} \\ C\left\{ \mathsf{aw}\!\downarrow \dfrac{\mathsf{f}}{a} \right\} \end{array}}}{C\{a\}}$$



- Instances of s where $\rho = \mathsf{aw}\!\uparrow$ remove edges. Furthermore, they reduce the size of the proof.

$$\mathsf{mc}{\downarrow}\,\frac{A \vee B}{C\left\{ \mathsf{aw}{\uparrow}\,\dfrac{a}{\mathsf{t}} \right\}} \longrightarrow \mathsf{mc}{\downarrow}\,\frac{\boxed{\begin{array}{c} A \\ \pi_A \,\Vert\, \{=,\mathsf{aw}{\downarrow}\} \\ C\left\{ \mathsf{aw}{\uparrow}\,\dfrac{a}{\mathsf{t}} \right\} \end{array}} \vee \boxed{\begin{array}{c} B \\ \pi_B \,\Vert\, \{=,\mathsf{aw}{\downarrow}\} \\ C\left\{ \mathsf{aw}{\uparrow}\,\dfrac{a}{\mathsf{t}} \right\} \end{array}}}{C\{\mathsf{t}\}}$$



- Instances of the trivial reduction i do not change the flow of the derivation and therefore cannot produce new cycles.

□

To eliminate all cycles from a derivation, one simply performs the procedure n times, once for each critical merge contraction.

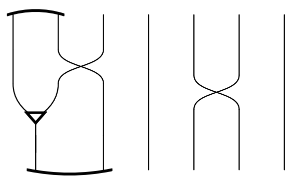
Theorem 4.17. *Given a derivation ϕ, there exists a derivation ψ with the same premiss and conclusion and without cycles.*

Proof. Given a derivation ϕ, we transform all its multicycles into unicycles by applying Lemma 4.4. If any two cycles with different identities share a cut, we apply Lemma 4.15. Lastly, we eliminate every critical merge contraction with an application of Theorem 4.16. □

4.3 Example

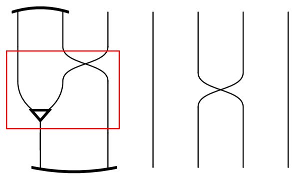
Example 4.18. We will remove the cycle in the following derivation:

$$
{\scriptstyle =}\,\dfrac{\boxed{{\scriptstyle m}\,\dfrac{{\scriptstyle \wedge\downarrow}\,\dfrac{\boxed{{\scriptstyle ai\downarrow}\,\dfrac{\mathsf{t}}{a\vee\bar a}}\wedge(C\wedge(a\vee B))}{(a\wedge C)\vee\boxed{{\scriptstyle =}\,\dfrac{\bar a\wedge(a\vee B)}{(a\vee B)\wedge\bar a}}}}{\boxed{{\scriptstyle =}\,\dfrac{a\vee(a\vee B)}{\boxed{{\scriptstyle ac\downarrow}\,\dfrac{a\vee a}{a}}\vee B}}\wedge(C\vee\bar a)}}\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{{\scriptstyle \wedge\downarrow}\,\dfrac{\boxed{{\scriptstyle \wedge\downarrow}\,\dfrac{(a\vee B)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}}\wedge\boxed{{\scriptstyle \wedge\downarrow}\,\dfrac{\boxed{{\scriptstyle =}\,\dfrac{C\vee\bar a}{\bar a\vee C}}\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{\boxed{{\scriptstyle s}\,\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{\boxed{{\scriptstyle ai\uparrow}\,\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}
$$



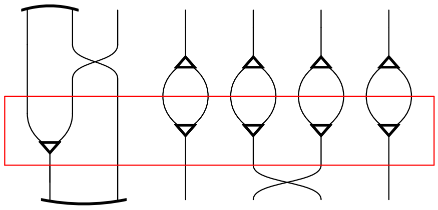
We find the lowest critical merge contraction, indicated in the derivation below by $\mathsf{mc}\downarrow$, and in the flow below by a red box.

$$= \dfrac{\boxed{\mathsf{mc}\!\downarrow \dfrac{\wedge\!\downarrow\dfrac{\boxed{ai\!\downarrow \dfrac{\mathsf{t}}{a\vee\bar a}} \wedge (C\wedge(a\vee B))}{(a\wedge C)\vee \boxed{= \dfrac{\bar a \wedge (a\vee B)}{(a\vee B)\wedge \bar a}}}}{(a\vee B)\wedge(C\vee\bar a)}} \wedge ((D_1\vee D_2)\wedge(E_1\vee E_2))}{\wedge\!\downarrow \dfrac{\boxed{\wedge\!\downarrow\dfrac{(a\vee B)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}} \wedge \boxed{\wedge\!\downarrow\dfrac{\boxed{=\dfrac{C\vee\bar a}{\bar a\vee C}}\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{\boxed{s\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{\boxed{ai\!\uparrow\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}$$



We apply an instance of the reduction t to permute past the equality rule.

$$\mathsf{mc}\!\downarrow\dfrac{s\dfrac{\boxed{\dfrac{\boxed{ai\!\downarrow \dfrac{\mathsf{t}}{a\vee\bar a}} \wedge (C\wedge(a\vee B))}{(a\wedge C)\vee((a\vee B)\wedge\bar a)}} \wedge \boxed{\mathsf{mc}\!\uparrow\dfrac{(D_1\vee D_2)\wedge(E_1\vee E_2)}{((D_1\vee D_2)\wedge(E_1\vee E_2))\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}}}{\boxed{=\dfrac{(a\wedge C)\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{(a\wedge(D_1\vee D_2))\wedge(C\wedge(E_1\vee E_2))}} \vee \boxed{=\dfrac{((a\vee B)\wedge\bar a)\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{((a\vee B)\wedge(D_1\vee D_2))\wedge(\bar a\wedge(E_1\vee E_2))}}}}{\wedge\!\downarrow \dfrac{\boxed{\wedge\!\downarrow\dfrac{(a\vee B)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}} \wedge \boxed{\wedge\!\downarrow\dfrac{\boxed{=\dfrac{C\vee\bar a}{\bar a\vee C}}\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{\boxed{s\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{\boxed{ai\!\uparrow\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}$$

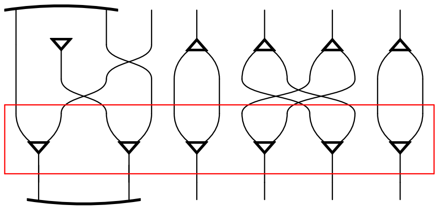


We apply an instance of s to permute past the left instance of $\wedge{\downarrow}$:

$$
\mathsf{mc}{\downarrow}\dfrac{s\dfrac{\boxed{\dfrac{\boxed{ai{\downarrow}\dfrac{\mathsf{t}}{a \vee \bar{a}}} \wedge (C \wedge (a \vee B))}{(a \wedge C) \vee ((a \vee B) \wedge \bar{a})}} \wedge \boxed{\mathsf{mc}{\uparrow}\dfrac{(D_1 \vee D_2) \wedge (E_1 \vee E_2)}{((D_1 \vee D_2) \wedge (E_1 \vee E_2)) \wedge ((D_1 \vee D_2) \wedge (E_1 \vee E_2))}}}{\boxed{=\dfrac{(a \wedge C) \wedge ((D_1 \vee D_2) \wedge (E_1 \vee E_2))}{\boxed{\wedge{\downarrow}\dfrac{\left(a \vee \boxed{w{\downarrow}\dfrac{\mathsf{f}}{B}}\right) \wedge (D_1 \vee D_2)}{(a \vee D_1) \vee (B \wedge D_2)}} \wedge (C \wedge (E_1 \vee E_2))}} \vee \boxed{=\dfrac{((a \vee B) \wedge \bar{a}) \wedge ((D_1 \vee D_2) \wedge (E_1 \vee E_2))}{\boxed{\wedge{\downarrow}\dfrac{(a \vee B) \wedge (D_1 \vee D_2)}{(a \vee D_1) \vee (B \wedge D_2)}} \wedge (\bar{a} \wedge (E_1 \vee E_2))}}}}{\wedge{\downarrow}\dfrac{((a \vee D_1) \vee (B \wedge D_2)) \wedge \boxed{\wedge{\downarrow}\dfrac{\boxed{=\dfrac{C \vee \bar{a}}{\bar{a} \vee C}} \wedge (E_1 \vee E_2)}{(\bar{a} \vee E_1) \vee (C \wedge E_2)}}}{\boxed{s\dfrac{(a \vee D_1) \wedge (\bar{a} \vee E_1)}{\boxed{ai{\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee (D_1 \vee E_1)}} \vee ((B \wedge D_2) \vee (C \wedge E_2))}}
$$

We apply two instances of s to permute past the commutativity rule and the rule $\wedge{\downarrow}$.

$$
\mathsf{mc}{\downarrow}\dfrac{s\dfrac{\boxed{\dfrac{\boxed{ai{\downarrow}\dfrac{\mathsf{t}}{a \vee \bar{a}}} \wedge (C \wedge (a \vee B))}{(a \wedge C) \vee ((a \vee B) \wedge \bar{a})}} \wedge \boxed{\mathsf{mc}{\uparrow}\dfrac{(D_1 \vee D_2) \wedge (E_1 \vee E_2)}{((D_1 \vee D_2) \wedge (E_1 \vee E_2)) \wedge ((D_1 \vee D_2) \wedge (E_1 \vee E_2))}}}{\boxed{=\dfrac{(a \wedge C) \wedge ((D_1 \vee D_2) \wedge (E_1 \vee E_2))}{\boxed{\wedge{\downarrow}\dfrac{\left(a \vee \boxed{w{\downarrow}\dfrac{\mathsf{f}}{B}}\right) \wedge (D_1 \vee D_2)}{(a \vee D_1) \vee (B \wedge D_2)}} \wedge \boxed{\wedge{\downarrow}\dfrac{\left(\boxed{w{\downarrow}\dfrac{\mathsf{f}}{\bar{a}}} \vee C\right) \wedge (E_1 \vee E_2)}{(\bar{a} \vee E_1) \vee (C \wedge E_2)}}}} \vee \boxed{=\dfrac{((a \vee B) \wedge \bar{a}) \wedge ((D_1 \vee D_2) \wedge (E_1 \vee E_2))}{\boxed{\wedge{\downarrow}\dfrac{(a \vee B) \wedge (D_1 \vee D_2)}{(a \vee D_1) \vee (B \wedge D_2)}} \wedge \boxed{\wedge{\downarrow}\dfrac{\left(\bar{a} \vee \boxed{w{\downarrow}\dfrac{\mathsf{f}}{C}}\right) \wedge (E_1 \vee E_2)}{(\bar{a} \vee E_1) \vee (C \wedge E_2)}}}}}}{\wedge{\downarrow}\dfrac{((a \vee D_1) \vee (B \wedge D_2)) \wedge ((\bar{a} \vee E_1) \vee (C \wedge E_2))}{\boxed{s\dfrac{(a \vee D_1) \wedge (\bar{a} \vee E_1)}{\boxed{ai{\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee (D_1 \vee E_1)}} \vee ((B \wedge D_2) \vee (C \wedge E_2))}}
$$



We apply an instance of s to permute past the rule $\wedge{\downarrow}$:

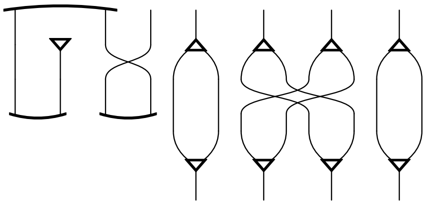
$$
s\,\dfrac{\boxed{\dfrac{\boxed{ai{\downarrow}\dfrac{\mathsf{t}}{a\vee\bar a}}\wedge(C\wedge(a\vee B))}{(a\wedge C)\vee((a\vee B)\wedge\bar a)}}\wedge\boxed{\mathsf{mc}{\uparrow}\dfrac{(D_1\vee D_2)\wedge(E_1\vee E_2)}{((D_1\vee D_2)\wedge(E_1\vee E_2))\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}}}{\mathsf{mc}{\downarrow}\dfrac{\boxed{=\dfrac{(a\wedge C)\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{\wedge{\downarrow}\dfrac{\boxed{\wedge{\downarrow}\dfrac{\left(a\vee\boxed{w{\downarrow}\dfrac{\mathsf{f}}{B}}\right)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}}\wedge\boxed{\wedge{\downarrow}\dfrac{\left(\boxed{w{\downarrow}\dfrac{\mathsf{f}}{\bar a}}\vee C\right)\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{((a\vee D_1)\wedge(\bar a\vee E_1))\vee((B\wedge D_2)\vee(C\wedge E_2))}}}\vee\boxed{=\dfrac{((a\vee B)\wedge\bar a)\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{\wedge{\downarrow}\dfrac{\boxed{\wedge{\downarrow}\dfrac{(a\vee B)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}}\wedge\boxed{\wedge{\downarrow}\dfrac{\left(\bar a\vee\boxed{w{\downarrow}\dfrac{\mathsf{f}}{C}}\right)\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{((a\vee D_1)\wedge(\bar a\vee E_1))\vee((B\wedge D_2)\vee(C\wedge E_2))}}}}{\boxed{s\,\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{\boxed{ai{\uparrow}\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}
$$

We then apply an instance of s to permute past the rule $\wedge{\downarrow}$:

$$
s\,\dfrac{\boxed{\dfrac{\boxed{ai{\downarrow}\dfrac{\mathsf{t}}{a\vee\bar a}}\wedge(C\wedge(a\vee B))}{(a\wedge C)\vee((a\vee B)\wedge\bar a)}}\wedge\boxed{\mathsf{mc}{\uparrow}\dfrac{(D_1\vee D_2)\wedge(E_1\vee E_2)}{((D_1\vee D_2)\wedge(E_1\vee E_2))\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}}}{\mathsf{mc}{\downarrow}\dfrac{\boxed{=\dfrac{(a\wedge C)\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{\wedge{\downarrow}\dfrac{\boxed{\wedge{\downarrow}\dfrac{\left(a\vee\boxed{w{\downarrow}\dfrac{\mathsf{f}}{B}}\right)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}}\wedge\boxed{\wedge{\downarrow}\dfrac{\left(\boxed{w{\downarrow}\dfrac{\mathsf{f}}{\bar a}}\vee C\right)\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{\boxed{\wedge{\downarrow}\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{(a\wedge\bar a)\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}}\vee\boxed{=\dfrac{((a\vee B)\wedge\bar a)\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{\wedge{\downarrow}\dfrac{\boxed{\wedge{\downarrow}\dfrac{(a\vee B)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}}\wedge\boxed{\wedge{\downarrow}\dfrac{\left(\bar a\vee\boxed{w{\downarrow}\dfrac{\mathsf{f}}{C}}\right)\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{\boxed{\wedge{\downarrow}\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{(a\wedge\bar a)\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}}}{\boxed{s\,\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{\boxed{ai{\uparrow}\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}
$$

Last, we apply an instance of s to permute past the cut:

$$
s\,\dfrac{\boxed{\dfrac{\boxed{ai{\downarrow}\dfrac{\mathsf{t}}{a\vee\bar a}}\wedge(C\wedge(a\vee B))}{(a\wedge C)\vee((a\vee B)\wedge\bar a)}}\wedge\boxed{\mathsf{mc}{\uparrow}\dfrac{(D_1\vee D_2)\wedge(E_1\vee E_2)}{((D_1\vee D_2)\wedge(E_1\vee E_2))\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}}}{\mathsf{mc}{\downarrow}\dfrac{\boxed{=\dfrac{(a\wedge C)\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{\wedge{\downarrow}\dfrac{\boxed{\wedge{\downarrow}\dfrac{\left(a\vee\boxed{w{\downarrow}\dfrac{\mathsf{f}}{B}}\right)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}}\wedge\boxed{\wedge{\downarrow}\dfrac{\left(\boxed{w{\downarrow}\dfrac{\mathsf{f}}{\bar a}}\vee C\right)\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{\boxed{\wedge{\downarrow}\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{\boxed{ai{\uparrow}\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}}\vee\boxed{=\dfrac{((a\vee B)\wedge\bar a)\wedge((D_1\vee D_2)\wedge(E_1\vee E_2))}{\wedge{\downarrow}\dfrac{\boxed{\wedge{\downarrow}\dfrac{(a\vee B)\wedge(D_1\vee D_2)}{(a\vee D_1)\vee(B\wedge D_2)}}\wedge\boxed{\wedge{\downarrow}\dfrac{\left(\bar a\vee\boxed{w{\downarrow}\dfrac{\mathsf{f}}{C}}\right)\wedge(E_1\vee E_2)}{(\bar a\vee E_1)\vee(C\wedge E_2)}}}{\boxed{\wedge{\downarrow}\dfrac{(a\vee D_1)\wedge(\bar a\vee E_1)}{\boxed{ai{\uparrow}\dfrac{a\wedge\bar a}{\mathsf{f}}}\vee(D_1\vee E_1)}}\vee((B\wedge D_2)\vee(C\wedge E_2))}}}}{(\mathsf{f}\vee(D_1\vee E_1))\vee((B\wedge D_2)\vee(C\wedge E_2))}}
$$



Chapter 5

Conclusion

In this thesis, we have achieved a series of technical results, by taking advantage of the generality provided by the subatomic methodology:

- We have provided a general characterisation of proof systems, in such a way that every rule is an instance of single, regular, linear, inference rule scheme. We showed how this characterisation encompasses such different systems as multiplicative additive linear logic, BV or classical logic, while remaining concise enough to be useful in generalising splitting and decomposition.
- We proved a generalised splitting theorem, allowing us to understand the properties of proof systems that the procedure hinges on. In this way, we prove cut-elimination for a whole class of substructural logics and show that splitting is a very general procedure that can be applied to many systems with any number of relations and units. Furthermore, we show that it is carried over by the identification of units, as happens in the case of BV. In addition, this generalisation provides useful guidelines for the design of linear proof systems, removing the search for cut-elimination from the design process.
- We have shown that the splitting procedure is not restricted to systems with binary connectives and can be extended to relations of different arities by proving a splitting theorem for SKV, a system with a modality.
- We have shown that admissibility is a property that goes beyond the cut-rule: as a corollary of splitting we have proved the admissibility of a whole class of rules that corresponds to those rules necessary to make the cut atomic, such as the rule $q\uparrow$ of BV or the associativity of $\wedge$ in classical logic.
- We provided general reduction rules for the permutation of generic contractions and cocontractions with other rules and a characterisation of the systems they can be applied to, including MALL and classical logic. By doing so, we showed that not only atomic contractions and cocontractions can be permuted downwards/upwards in a derivation, but that in fact it is possible to permute a whole class of rules. The ability to permute atomic contractions and cocontractions in MALL and

classical logic is an instance of this phenomenon, and is due to certain properties that both systems share.

- We used the general reduction rules to design a procedure to remove ai-cycles in SKS and SMALLS proofs, proving the independence of the decomposition procedure from cut-elimination, and advancing towards being able to ascertain the complexity cost of the removal of cycles.

These results leave room for future developments, some of which are currently being researched:

- It would be interesting to provide a characterisation of sound rules in terms of an order between the relations: the design of systems would be much simplified, and the characterisation of systems would be further improved, maintaining the properties of the characterisation we provided in this work while gaining in specificity.

- Generalising the characterisation of rules and the splitting result to relations of different arities to include modalities and exponentials is expected to be a close future development, since the study of the deep inference systems for linear logic (with exponentials) [41], for classical predicate logic [4] or for BV has yielded very encouraging results towards the characterisation of the rules involving the exponentials with a single shape.

- Obtaining full decomposition for classical logic and for MALL in such a way that we can rewrite proofs into a splittable phase followed by a contractive phase is now a matter of finding the correct measure to prove that the permutations of generic contractions terminate.

- The removal of cycles from proofs has been proved to be a quadratic-time procedure in the sequent calculus [12]. By studying the procedure presented in this thesis, it will be possible to understand the complexity cost of cycle removal in deep inference.

The characterisation of rules through a single inference rule scheme was initially intended as a stepping stone towards the development of a graphical formalism that could be used to represent a wide variety of logics. The task however proved to be more daunting than we expected: to develop this formalism, a full understanding of the properties required for the normalisation procedures that we want to capture to isolate the complexity generating mechanisms (cut-elimination and decomposition) proved to be necessary. For that, a refinement of the general rule scheme was needed, and so the development of conditions on the relations that enable us to capture the normalisation procedures while maintaining generality came about. This characterisation was no easy task, since it needs to encompass both the linear and the contractive rules, that vary in behaviour and in shape in different non-subatomic systems.

Once the adequate characterisation was found, we proceeded to study cut-elimination and decomposition with this new methodology, with a strong focus on understanding the properties of the rules that are essential to obtain them. The generalisations of both of these procedures highlight which features should be captured by a graphical formalism: duality and contractiveness. When the final missing feature consisting of the extension of the notion of rank of an atomic contraction to generic contractions is found, we will have a description of all the elements that need to be featured in a graphical formalism in which cut-elimination and decomposition are naturally represented. I would very much like to continue towards this research direction: this thesis is a good start that provides many of the tools that I expect to use.

In short, in this work we have uncovered an underlying structure behind the shape of inference rules. This observation is truly surprising, and its generality can be exploited in many ways. Here, we used it to characterise proof systems and to study normalisation procedures, and it is expected that in the future the number of applications will only grow.

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