1. (a) see notes

(b) Let Π_i be proofs that select a conjunction component, like:

$$\begin{array}{rc}
 & C_1 \wedge \cdots \wedge C_k \\ \hline
\wedge_{ER} & C_2 \wedge \cdots \wedge C_k \\ \hline
\wedge_{ER} & \vdots \\ \hline
\wedge_{ER} & C_i \wedge C_{i+1} \wedge \cdots \wedge C_k \\ \hline
\wedge_{EL} & C_i \wedge C_{i+1} \wedge \cdots \wedge C_{k-1} \\ \hline
\wedge_{EL} & \vdots \\ \hline
 & C_i
\end{array}$$

assuming an unfavourable associativity of $\wedge$'s. We use Π_i in the following, where

$$C_n = (p_0 \vee q_0) \wedge A_1 \wedge B_1 \wedge \cdots \wedge A_n \wedge B_n :$$

$$\dfrac{\begin{matrix}[C_n]\\ \Pi_1 \\ p_0 \vee q_0\end{matrix} \qquad \dfrac{[p_0] \quad \begin{matrix}[C_n]\\ \Pi_2 \\ A_1 \\ \| \\ p_0 \to (p_1 \vee q_1)\end{matrix}}{p_1 \vee q_1}\to_E \qquad \dfrac{[q_0] \quad \begin{matrix}[C_n]\\ \Pi_3 \\ B_1 \\ \| \\ q_0 \to (p_1 \vee q_1)\end{matrix}}{p_1 \vee q_1}\to_E}{p_1 \vee q_1}\vee_E$$

same

$$\cdots$$

$$\dfrac{\dfrac{p_{n-1} \vee q_{n-1} \qquad \dfrac{[p_{n-1}] \quad \begin{matrix}[C_n]\\ \Pi_{2n} \\ A_n \\ \| \\ p_{n-1} \to (p_n \vee q_n)\end{matrix}}{p_n \vee q_n}\to_E \qquad \dfrac{[q_{n-1}] \quad \begin{matrix}[C_n]\\ \Pi_{2n+1} \\ B_n \\ \| \\ q_{n-1} \to (p_n \vee q_n)\end{matrix}}{p_n \vee q_n}\to_E}{p_n \vee q_n}}{C_n \to (p_n \vee q_n)}$$

$$\|$$

$$F_n$$

5 + 8 ← use of side conditions

2. (a) See notes

(b)

$$
\begin{array}{rc}
\cline{2-2}
 & p(w) \vdash p(w) \\
\cline{2-2}
<_L; <_L; <_L; <_R & p(x), q(z), p(w), q(z) \vdash p(w), p(y') \\
\cline{2-2}
\to_R; \to_R & p(x), q(z) \vdash p(w),\ p(w) \to (q(z) \to p(y')) \\
\cline{2-2}
\forall_R; \exists_R & p(x), q(z) \vdash p(w),\ \forall y. \exists z. (p(w) \to (q(z) \to p(y))) \\
\cline{2-2}
\exists_R & p(x), q(z) \vdash p(w),\ \exists x. \forall y. \exists z. (p(x) \to (q(z) \to p(y))) \\
\cline{2-2}
\to_R; \to_R & \vdash p(x) \to (q(z) \to p(w)),\ \exists x. \forall y. \exists z. (p(x) \to (q(z) \to p(y))) \\
\cline{2-2}
\exists_R & \vdash \exists z. (p(x) \to (q(z) \to p(w))),\ \exists x. \forall y. \exists z. (p(x) \to (q(z) \to p(y))) \\
\cline{2-2}
\forall_R & \vdash \forall y. \exists z. (p(x) \to (q(z) \to p(y))),\ \exists x. \forall y. \exists z. (p(x) \to (q(z) \to p(y))) \\
\cline{2-2}
\exists_R & \vdash \exists x. \forall y. \exists z. (p(x) \to (q(z) \to p(y))),\ \exists x. \forall y. \exists z. (p(x) \to (q(z) \to p(y))) \\
\cline{2-2}
>_R & \vdash \exists x. \forall y. \exists z. (p(x) \to (q(z) \to p(y)))
\end{array}
$$

3. (a) See notes.

(b)

$$\dfrac{\dfrac{\dfrac{\overline{A\vdash A}}{A,B\vdash A}\,{<_L}}{A\vdash B\to A}\,{\to_R}}{\vdash A\to(B\to A)}\,{\to_R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A\vdash A}}{A\to B, A\vdash A}\,{<_L} \qquad \dfrac{\dfrac{\dfrac{\overline{A\vdash A}}{A\vdash B,A}\,{<_R} \quad \dfrac{\overline{B\vdash B}}{B,A\vdash B}\,{<_L}}{A\to B,\ A\vdash B} \quad \dfrac{\overline{C\vdash C}}{C,\ A\to B, A\vdash C}\,{<_L^2}}{B\to C,\ A\to B,\ A\vdash C}}{A\to(B\to C),\ A\to B,\ A\vdash C}}{A\to(B\to C),\ A\to B\vdash A\to C}\,{\to_R}}{A\to(B\to C)\vdash (A\to B)\to(A\to C)}\,{\to_R}}{\vdash (A\to(B\to C))\to((A\to B)\to(A\to C))}\,{\to_R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{B\vdash B}}{\vdash B,\neg B}\,{\neg_R}}{\neg B\to A\vdash B,\neg B}\,{<_L} \qquad \dfrac{\dfrac{\dfrac{\overline{B\vdash B}}{\vdash \neg B, B}\,{\neg_R}}{\neg A\vdash \neg B, B}\,{<_L} \quad \dfrac{\dfrac{\overline{A\vdash A}}{\neg A, A\vdash}\,{\neg_L}}{\neg A, A\vdash B}\,{<_R}}{\neg A,\ \neg B\to A\vdash B}}{\neg B\to\neg A,\ \neg B\to A\ \vdash B}}{\neg B\to\neg A\ \vdash\ (\neg B\to A)\to B}\,{\to_L}}{\vdash (\neg B\to\neg A)\to((\neg B\to A)\to B)}\,{\to_L}$$

(c) Given a proof Π in HT, we reason by induction, where

$$\Pi = \ mp\ \dfrac{\begin{matrix}\Pi' \\ \vdots \\ A\end{matrix} \qquad \begin{matrix}\Pi'' \\ \vdots \\ A\to B\end{matrix}}{B}$$

in the inductive case (the base cases have been dealt with above).

Let Π'^{*} and Π''^{*} be proofs in LK obtained from Π' and Π'', we build

$$\dfrac{\begin{matrix}\Pi'^{*} \\ \vdots \\ \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\Pi''^{*} \\ \vdots \\ \vdash A\to B\end{matrix} \qquad \dfrac{\dfrac{\overline{A\vdash A}}{A\vdash B,A}\,{<_R} \quad \dfrac{\overline{B\vdash B}}{B,A\vdash B}\,{<_L}}{A\to B,\ A\vdash B}}{A\vdash B}\,{\bowtie}}{\vdash B}\,{\bowtie}$$

4. (a) See notes

(b) 1. $x \models \Diamond\neg A$ iff $\exists y.\ (Rxy \wedge y \models \neg A)$

iff $\neg\forall y.\ (Rxy \Rightarrow y \models A)$

iff $x \models \neg\Box A$

2. $x \models \Box\neg A$ iff $\forall y.\ (Rxy \Rightarrow y \models \neg A)$ (De Morgan dual)

iff $\neg\exists y.\ (Rxy \wedge y \models A)$

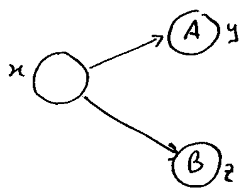
iff $x \models \neg\Diamond A$

3. $x \models \Box(A \wedge B)$ iff $\forall y.\ (Rxy \Rightarrow (y \models A \wedge y \models B))$

iff $\forall y\ (Rxy \Rightarrow y \models A) \wedge \forall y\ (Rxy \Rightarrow y \models B)$

iff $x \models \Box A \wedge \Box B$

4. counterexample:



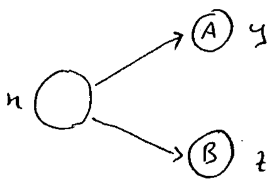
$x \models \Box(A \vee B)$

but

$x \not\models \Box A$

$x \not\models \Box B$

5. De Morgan dual of 4; counterexample:



$x \models \Diamond A \wedge \Diamond B$

but

$x \not\models \Diamond(A \wedge B)$

6. De Morgan dual of 3:

$x \models \Diamond(A \vee B)$ iff $\exists y.\ (Rxy \wedge (y \models A \vee y \models B))$

iff $\exists y.\ ((Rxy \wedge y \models A) \vee (Rxy \wedge y \models B))$

iff $\exists y.\ (Rxy \wedge y \models A) \vee \exists y\ (Rxy \wedge y \models B)$

iff $x \models \Diamond A \vee \Diamond B$