Introduction to Sequent Calculus and Abstract Logic Programming

Homework Exercises

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1 Homework 1

Find proofs in $\mathsf{N}_{\mathsf{I}}^{\supset,\wedge,\vee,\bot}$ or $\mathsf{N}_{\mathsf{C}}^{\supset,\wedge,\vee,\bot}$ of the following formulae:

1) $(A \wedge (A \supset B)) \supset B$;

2) $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$;

3) $((A \supset B) \supset A) \supset A$;

4) $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$;

5) $A \supset ((A \wedge B) \vee (A \wedge \neg B))$.

2 Solutions to Homework 1

Please keep in mind that there are several possible ways of proving formulae. The ones you find here are just suggestions.

For some of the following proofs, it is convenient to consider separately a deduction of the formula $A \vee \neg A$ in the hypotheses Γ. Let $\Delta(\Gamma, A)$ be the deduction

$$
\mathsf{bc}\,\dfrac{\supset_{\mathsf{E}}\dfrac{\Gamma_1 \vdash \neg(A\vee\neg A) \qquad \vee_{\mathsf{IR}}\dfrac{\supset_{\mathsf{I}}\dfrac{\supset_{\mathsf{E}}\dfrac{\Gamma_1, y\colon A \vdash \neg(A\vee\neg A) \qquad \vee_{\mathsf{IL}}\dfrac{\Gamma_1,y\colon A\vdash A}{\Gamma_1,y\colon A\vdash A\vee\neg A}}{\Gamma_1,y\colon A\vdash\bot}}{\Gamma_1\vdash\neg A}}{\Gamma_1\vdash A\vee\neg A}}{\Gamma,x\colon\neg(A\vee\neg A)\vdash\bot}}{\Gamma\vdash A\vee\neg A}
$$

where $\Gamma_1 = \Gamma \cup \{x\colon \neg(A \vee \neg A)\}$.

1)

$$
\supset_{\mathsf{I}}\dfrac{\supset_{\mathsf{E}}\dfrac{\wedge_{\mathsf{ER}}\dfrac{x\colon A\wedge(A\supset B)\vdash A\wedge(A\supset B)}{x\colon A\wedge(A\supset B)\vdash A\supset B} \qquad \wedge_{\mathsf{EL}}\dfrac{x\colon A\wedge(A\supset B)\vdash A\wedge(A\supset B)}{x\colon A\wedge(A\supset B)\vdash A}}{x\colon A\wedge(A\supset B)\vdash B}}{\vdash (A\wedge(A\supset B))\supset B}
$$

2a)

$$
\supset_{\mathsf{I}}\dfrac{\vee_{\mathsf{E}}\dfrac{\wedge_{\mathsf{ER}}\dfrac{\Gamma\vdash A\wedge(B\vee C)}{\Gamma\vdash B\vee C} \qquad \vee_{\mathsf{IL}}\dfrac{\wedge_{\mathsf{I}}\dfrac{\wedge_{\mathsf{EL}}\dfrac{\Gamma_1\vdash A\wedge(B\vee C)}{\Gamma_1\vdash A} \qquad \Gamma_1\vdash B}{\Gamma_1\vdash A\wedge B}}{\Gamma_1\vdash (A\wedge B)\vee(A\wedge C)} \qquad \vee_{\mathsf{IR}}\dfrac{\wedge_{\mathsf{I}}\dfrac{\wedge_{\mathsf{EL}}\dfrac{\Gamma_2\vdash A\wedge(B\vee C)}{\Gamma_2\vdash A} \qquad \Gamma_2\vdash C}{\Gamma_2\vdash A\wedge C}}{\Gamma_2\vdash (A\wedge B)\vee(A\wedge C)}}{x\colon A\wedge(B\vee C)\vdash (A\wedge B)\vee(A\wedge C)}}{\vdash (A\wedge(B\vee C))\supset((A\wedge B)\vee(A\wedge C))}
$$

where $\Gamma = \{x\colon A \wedge (B \vee C)\}$, $\Gamma_1 = \Gamma \cup \{y\colon B\}$ and $\Gamma_2 = \Gamma \cup \{z\colon C\}$.

2b)

$$
{\scriptstyle\supset_{\mathsf I}}\dfrac{{\scriptstyle\vee_{\mathsf E}}\dfrac{\Gamma\vdash(A\wedge B)\vee(A\wedge C)\qquad {\scriptstyle\wedge_{\mathsf I}}\dfrac{{\scriptstyle\wedge_{\mathsf{EL}}}\dfrac{\Gamma_1\vdash A\wedge B}{\Gamma_1\vdash A}\quad {\scriptstyle\vee_{\mathsf{IL}}}\dfrac{{\scriptstyle\wedge_{\mathsf{ER}}}\dfrac{\Gamma_1\vdash A\wedge B}{\Gamma_1\vdash B}}{\Gamma_1\vdash B\vee C}}{\Gamma_1\vdash A\wedge(B\vee C)}\qquad {\scriptstyle\wedge_{\mathsf I}}\dfrac{{\scriptstyle\wedge_{\mathsf{EL}}}\dfrac{\Gamma_2\vdash A\wedge C}{\Gamma_2\vdash A}\quad {\scriptstyle\vee_{\mathsf{IR}}}\dfrac{{\scriptstyle\wedge_{\mathsf{ER}}}\dfrac{\Gamma_2\vdash A\wedge C}{\Gamma_2\vdash C}}{\Gamma_2\vdash B\vee C}}{\Gamma_2\vdash A\wedge(B\vee C)}}{x\colon(A\wedge B)\vee(A\wedge C)\vdash A\wedge(B\vee C)}}{\vdash((A\wedge B)\vee(A\wedge C))\supset(A\wedge(B\vee C))}
$$

where $\Gamma = \{x\colon (A \wedge B) \vee (A \wedge C)\}$, $\Gamma_1 = \Gamma \cup \{y\colon A \wedge B\}$ and $\Gamma_2 = \Gamma \cup \{z\colon A \wedge C\}$.

3)

$$
{\scriptstyle\supset_{\mathsf I}}\dfrac{{\scriptstyle\vee_{\mathsf E}}\dfrac{\genfrac{}{}{0pt}{}{\Delta(\Gamma,A\supset B)}{\Gamma\vdash(A\supset B)\vee\neg(A\supset B)}\qquad {\scriptstyle\supset_{\mathsf E}}\dfrac{\Gamma_1\vdash(A\supset B)\supset A\quad \Gamma_1\vdash A\supset B}{\Gamma_1\vdash A}\qquad {\scriptstyle\mathsf{bc}}\dfrac{{\scriptstyle\supset_{\mathsf E}}\dfrac{\Gamma_3\vdash\neg(A\supset B)\quad {\scriptstyle\supset_{\mathsf I}}\dfrac{{\scriptstyle\mathsf{bc}}\dfrac{{\scriptstyle\supset_{\mathsf E}}\dfrac{\Gamma_5\vdash\neg A\quad \Gamma_5\vdash A}{\Gamma_5\vdash\bot}}{\Gamma_4\vdash B}}{\Gamma_3\vdash A\supset B}}{\Gamma_3\vdash\bot}}{\Gamma_2\vdash A}}{x\colon(A\supset B)\supset A\vdash A}}{\vdash((A\supset B)\supset A)\supset A}
$$

where $\Gamma = \{x\colon (A \supset B) \supset A\}$, $\Gamma_1 = \Gamma \cup \{y\colon A \supset B\}$, $\Gamma_2 = \Gamma \cup \{z\colon \neg(A \supset B)\}$, $\Gamma_3 = \Gamma_2 \cup \{t\colon \neg A\}$, $\Gamma_4 = \Gamma_3 \cup \{u\colon A\}$ and $\Gamma_5 = \Gamma_4 \cup \{v\colon \neg B\}$.

4a)

$$
{\scriptstyle\supset_{\mathsf I}}\dfrac{{\scriptstyle\wedge_{\mathsf I}}\dfrac{{\scriptstyle\vee_{\mathsf E}}\dfrac{\Gamma\vdash A\vee(B\wedge C)\quad {\scriptstyle\vee_{\mathsf{IL}}}\dfrac{\Gamma_1\vdash A}{\Gamma_1\vdash A\vee B}\quad {\scriptstyle\vee_{\mathsf{IR}}}\dfrac{{\scriptstyle\wedge_{\mathsf{EL}}}\dfrac{\Gamma_2\vdash B\wedge C}{\Gamma_2\vdash B}}{\Gamma_2\vdash A\vee B}}{\Gamma\vdash A\vee B}\qquad {\scriptstyle\vee_{\mathsf E}}\dfrac{\Gamma\vdash A\vee(B\wedge C)\quad {\scriptstyle\vee_{\mathsf{IL}}}\dfrac{\Gamma_1\vdash A}{\Gamma_1\vdash A\vee C}\quad {\scriptstyle\vee_{\mathsf{IR}}}\dfrac{{\scriptstyle\wedge_{\mathsf{ER}}}\dfrac{\Gamma_2\vdash B\wedge C}{\Gamma_2\vdash C}}{\Gamma_2\vdash A\vee C}}{\Gamma\vdash A\vee C}}{x\colon A\vee(B\wedge C)\vdash(A\vee B)\wedge(A\vee C)}}{\vdash(A\vee(B\wedge C))\supset((A\vee B)\wedge(A\vee C))}
$$

where $\Gamma = \{x\colon A \vee (B \wedge C)\}$, $\Gamma_1 = \Gamma \cup \{y\colon A\}$ and $\Gamma_2 = \Gamma \cup \{z\colon B \wedge C\}$.

4b)

$$
{\scriptstyle\supset_{\mathsf I}}\dfrac{{\scriptstyle\vee_{\mathsf E}}\dfrac{\genfrac{}{}{0pt}{}{\Delta(\Gamma,A)}{\Gamma\vdash A\vee\neg A}\quad {\scriptstyle\vee_{\mathsf{IL}}}\dfrac{\Gamma_1\vdash A}{\Gamma_1\vdash A\vee(B\wedge C)}\quad {\scriptstyle\vee_{\mathsf{IR}}}\dfrac{{\scriptstyle\wedge_{\mathsf I}}\dfrac{{\scriptstyle\vee_{\mathsf E}}\dfrac{{\scriptstyle\wedge_{\mathsf{EL}}}\dfrac{\Gamma_2\vdash(A\vee B)\wedge(A\vee C)}{\Gamma_2\vdash A\vee B}\quad {\scriptstyle\mathsf{bc}}\dfrac{{\scriptstyle\supset_{\mathsf E}}\dfrac{\Gamma_5\vdash\neg A\quad \Gamma_5\vdash A}{\Gamma_5\vdash\bot}}{\Gamma_3\vdash B}\quad \Gamma_4\vdash B}{\Gamma_2\vdash B}\qquad {\scriptstyle\vee_{\mathsf E}}\dfrac{{\scriptstyle\wedge_{\mathsf{ER}}}\dfrac{\Gamma_2\vdash(A\vee B)\wedge(A\vee C)}{\Gamma_2\vdash A\vee C}\quad {\scriptstyle\mathsf{bc}}\dfrac{{\scriptstyle\supset_{\mathsf E}}\dfrac{\Gamma'_5\vdash\neg A\quad \Gamma'_5\vdash A}{\Gamma'_5\vdash\bot}}{\Gamma_3\vdash C}\quad \Gamma'_4\vdash C}{\Gamma_2\vdash C}}{\Gamma_2\vdash B\wedge C}}{\Gamma_2\vdash A\vee(B\wedge C)}}{x\colon(A\vee B)\wedge(A\vee C)\vdash A\vee(B\wedge C)}}{\vdash((A\vee B)\wedge(A\vee C))\supset(A\vee(B\wedge C))}
$$

where $\Gamma = \{x\colon (A\vee B)\wedge(A\vee C)\}$, $\Gamma_1 = \Gamma\cup\{y\colon A\}$, $\Gamma_2 = \Gamma\cup\{z\colon \neg A\}$, $\Gamma_3 = \Gamma_2\cup\{t\colon A\}$, $\Gamma_4 = \Gamma_2\cup\{u\colon B\}$, $\Gamma'_4 = \Gamma_2 \cup \{u'\colon C\}$, $\Gamma_5 = \Gamma_3 \cup \{v\colon \neg B\}$ and $\Gamma'_5 = \Gamma_3 \cup \{v'\colon \neg C\}$.

5)

$$
{\scriptstyle\supset_{\mathsf I}}\dfrac{{\scriptstyle\vee_{\mathsf E}}\dfrac{\genfrac{}{}{0pt}{}{\Delta(\{x\colon A\},B)}{x\colon A\vdash B\vee\neg B}\quad {\scriptstyle\vee_{\mathsf{IL}}}\dfrac{{\scriptstyle\wedge_{\mathsf I}}\dfrac{x\colon A,y\colon B\vdash A\quad x\colon A,y\colon B\vdash B}{x\colon A,y\colon B\vdash A\wedge B}}{x\colon A,y\colon B\vdash(A\wedge B)\vee(A\wedge\neg B)}\quad {\scriptstyle\vee_{\mathsf{IR}}}\dfrac{{\scriptstyle\wedge_{\mathsf I}}\dfrac{x\colon A,z\colon\neg B\vdash A\quad x\colon A,z\colon\neg B\vdash\neg B}{x\colon A,z\colon\neg B\vdash A\wedge\neg B}}{x\colon A,z\colon\neg B\vdash(A\wedge B)\vee(A\wedge\neg B)}}{x\colon A\vdash(A\wedge B)\vee(A\wedge\neg B)}}{\vdash A\supset((A\wedge B)\vee(A\wedge\neg B))}
$$

3 Homework 2

1) Prove that $\mathsf{G}_\mathsf{c}^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$ and LK are equivalent by giving two effective procedures which transform derivations in one system to equivalent derivations in the other, and vice-versa.

2) Prove $A \vee \neg A$ both in $\mathsf{G}_\mathsf{c}^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$ and LK.

3) Prove $((A \supset B) \supset A) \supset A$ both in $\mathsf{G}_\mathsf{c}^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$ and LK.

4) Prove in LK the formula $\forall x.(A \supset B) \supset (A \supset \forall x.B)$, where x is not free in A.

4 Solutions to Homework 2

1a) Given a proof Π in $\mathsf{G}_\mathsf{c}^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$, we will build a proof $f(\Pi)$ in LK whose conclusion is the same. We shall proceed by structural induction on Π.

Base Case

$$\Pi = A, \Gamma \vdash \Delta, A \qquad\qquad f(\Pi) = \begin{array}{rc} & A \vdash A \\ \cline{2-2} <_\mathsf{L} & \vdots \\ \cline{2-2} <_\mathsf{L} & A, \Gamma \vdash A \\ \cline{2-2} <_\mathsf{R} & \vdots \\ \cline{2-2} <_\mathsf{R} & A, \Gamma \vdash \Delta, A \end{array}$$

Inductive Cases The only interesting cases occur when the root rule in Π is either $\bowtie$, $\wedge_\mathsf{L}$ or $\vee_\mathsf{R}$:

$$\Pi = \begin{array}{rcc} & \Pi_1 & \Pi_2 \\ & \Gamma \vdash \Delta, A & A, \Gamma \vdash \Delta \\ \cline{2-3} \bowtie & \multicolumn{2}{c}{\Gamma \vdash \Delta} \end{array} \qquad\qquad f(\Pi) = \begin{array}{rcc} & f(\Pi_1) & f(\Pi_2) \\ & \Gamma \vdash \Delta, A & A, \Gamma \vdash \Delta \\ \cline{2-3} \bowtie & \multicolumn{2}{c}{\Gamma, \Gamma \vdash \Delta, \Delta} \\ \cline{2-3} >_\mathsf{L} & \multicolumn{2}{c}{\vdots} \\ \cline{2-3} >_\mathsf{L} & \multicolumn{2}{c}{\Gamma \vdash \Delta, \Delta} \\ \cline{2-3} >_\mathsf{R} & \multicolumn{2}{c}{\vdots} \\ \cline{2-3} >_\mathsf{R} & \multicolumn{2}{c}{\Gamma \vdash \Delta} \end{array}$$

$$\Pi = \begin{array}{rc} & \Pi' \\ & A, B, \Gamma \vdash \Delta \\ \cline{2-2} \wedge_\mathsf{L} & A \wedge B, \Gamma \vdash \Delta \end{array} \qquad\qquad f(\Pi) = \begin{array}{rc} & f(\Pi') \\ & A, B, \Gamma \vdash \Delta \\ \cline{2-2} \wedge_\mathsf{LL} & A \wedge B, B, \Gamma \vdash \Delta \\ \cline{2-2} \wedge_\mathsf{LR} & A \wedge B, A \wedge B, \Gamma \vdash \Delta \\ \cline{2-2} >_\mathsf{L} & A \wedge B, \Gamma \vdash \Delta \end{array}$$

$$\Pi = \begin{array}{c}\Pi' \\ \vee_{\mathsf{R}}\dfrac{\Gamma\vdash\Delta,A,B}{\Gamma\vdash\Delta,A\vee B}\end{array} \qquad f(\Pi) = \begin{array}{c} f(\Pi') \\ \vee_{\mathsf{RR}}\dfrac{\vee_{\mathsf{RL}}\dfrac{\Gamma\vdash\Delta,A,B}{\Gamma\vdash\Delta,A\vee B,B}}{>_{\mathsf{R}}\dfrac{\Gamma\vdash\Delta,A\vee B,A\vee B}{\Gamma\vdash\Delta,A\vee B}}\end{array}$$

1b) Given a proof Π in $\mathsf{G}_{\mathsf{c}}^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$, let us define $\Gamma\to\Pi$ (respectively, $\Pi\leftarrow\Gamma$) as the proof one obtains by adding Γ to the left side (right side) in each sequent appearing in Π, where Γ is a multiset of formulae. Since in this way eigenvariables may be captured, they have to be renamed to some variable not appearing free in Γ.
Given a proof Π in LK, we will build a proof $g(\Pi)$ in $\mathsf{G}_{\mathsf{c}}^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$ whose conclusion is the same. We shall proceed by structural induction on Π.

Base Case

$$\Pi = A\vdash A \qquad\qquad g(\Pi) = A\vdash A$$

Inductive Cases The only interesting cases occur when the root rule in Π is either $\bowtie$, $<_{\mathsf{L}}$, $<_{\mathsf{R}}$, $\wedge_{\mathsf{LL}}$, $\wedge_{\mathsf{LR}}$, $\vee_{\mathsf{RL}}$ or $\vee_{\mathsf{RR}}$:

$$\Pi = \begin{array}{c}\Pi_1 \qquad \Pi_2 \\ \bowtie\dfrac{\Gamma\vdash\Delta,A \quad A,\Lambda\vdash\Theta}{\Gamma,\Lambda\vdash\Delta,\Theta}\end{array} \qquad g(\Pi) = \begin{array}{c}\Lambda{\to}g(\Pi_1){\leftarrow}\Theta \qquad\qquad \Gamma{\to}g(\Pi_2){\leftarrow}\Delta \\ \bowtie\dfrac{\Gamma,\Lambda\vdash\Delta,\Theta,A \qquad\qquad A,\Gamma,\Lambda\vdash\Delta,\Theta}{\Gamma,\Lambda\vdash\Delta,\Theta}\end{array}$$

$$\Pi = \begin{array}{c}\Pi' \\ <_{\mathsf{L}}\dfrac{\Gamma\vdash\Delta}{A,\Gamma\vdash\Delta}\end{array} \qquad g(\Pi) = \{A\}_+\to g(\Pi')$$

$$\Pi = \begin{array}{c}\Pi' \\ <_{\mathsf{R}}\dfrac{\Gamma\vdash\Delta}{\Gamma\vdash\Delta,A}\end{array} \qquad g(\Pi) = g(\Pi')\leftarrow\{A\}_+$$

$$\Pi = \begin{array}{c}\Pi' \\ \wedge_{\mathsf{LL}}\dfrac{A,\Gamma\vdash\Delta}{A\wedge B,\Gamma\vdash\Delta}\end{array} \qquad g(\Pi) = \begin{array}{c}\{B\}_+{\to}g(\Pi') \\ \wedge_{\mathsf{L}}\dfrac{A,B,\Gamma\vdash\Delta}{A\wedge B,\Gamma\vdash\Delta}\end{array}$$

$$\Pi = \quad \wedge_{\mathsf{LR}}\,\dfrac{\begin{matrix}\Pi' \\ B, \varGamma \vdash \varDelta\end{matrix}}{A \wedge B, \varGamma \vdash \varDelta} \qquad\qquad g(\Pi) = \quad \wedge_{\mathsf{L}}\,\dfrac{\begin{matrix}\{A\}_{+}\!\rightarrow g(\Pi') \\ A, B, \varGamma \vdash \varDelta\end{matrix}}{A \wedge B, \varGamma \vdash \varDelta}$$

$$\Pi = \quad \vee_{\mathsf{RL}}\,\dfrac{\begin{matrix}\Pi' \\ \varGamma \vdash \varDelta, A\end{matrix}}{\varGamma \vdash \varDelta, A \vee B} \qquad\qquad g(\Pi) = \quad \vee_{\mathsf{R}}\,\dfrac{\begin{matrix}g(\Pi')\!\leftarrow\{B\}_{+} \\ \varGamma \vdash \varDelta, A, B\end{matrix}}{\varGamma \vdash \varDelta, A \vee B}$$

$$\Pi = \quad \vee_{\mathsf{RR}}\,\dfrac{\begin{matrix}\Pi' \\ \varGamma \vdash \varDelta, B\end{matrix}}{\varGamma \vdash \varDelta, A \vee B} \qquad\qquad g(\Pi) = \quad \vee_{\mathsf{R}}\,\dfrac{\begin{matrix}g(\Pi')\!\leftarrow\{A\}_{+} \\ \varGamma \vdash \varDelta, A, B\end{matrix}}{\varGamma \vdash \varDelta, A \vee B}$$

2a) In $\mathsf{G}_{\mathsf{c}}^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$:

$$\vee_{\mathsf{R}}\,\dfrac{\neg_{\mathsf{R}}\,\dfrac{A \vdash A}{\vdash A, \neg A}}{\vdash A \vee \neg A}$$

2b) In LK:

$$>_{\mathsf{R}}\,\dfrac{\vee_{\mathsf{RL}}\,\dfrac{\vee_{\mathsf{RR}}\,\dfrac{\neg_{\mathsf{R}}\,\dfrac{A \vdash A}{\vdash A, \neg A}}{\vdash A, A \vee \neg A}}{\vdash A \vee \neg A, A \vee \neg A}}{\vdash A \vee \neg A}$$

3a) In $\mathsf{G}_{\mathsf{c}}^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$:

$$\supset_{\mathsf{R}}\,\dfrac{\supset_{\mathsf{L}}\,\dfrac{\supset_{\mathsf{R}}\,\dfrac{A \vdash B, A}{\vdash A \supset B, A} \quad A \vdash A}{(A \supset B) \supset A \vdash A}}{\vdash ((A \supset B) \supset A) \supset A}$$

3b) In LK:

$$\supset_{\mathsf{R}}\,\dfrac{\supset_{\mathsf{L}}\,\dfrac{\supset_{\mathsf{R}}\,\dfrac{<_{\mathsf{R}}\,\dfrac{A \vdash A}{A \vdash B, A}}{\vdash A \supset B, A} \quad A \vdash A}{(A \supset B) \supset A \vdash A}}{\vdash ((A \supset B) \supset A) \supset A}$$

4) In LK, where x is not free in A:

$$\supset_{\mathsf{R}}\dfrac{\supset_{\mathsf{R}}\dfrac{\forall_{\mathsf{R}}\dfrac{\forall_{\mathsf{L}}\dfrac{\supset_{\mathsf{L}}\dfrac{<_{\mathsf{R}}\dfrac{A \vdash A}{A \vdash A, B} \quad <_{\mathsf{L}}\dfrac{B \vdash B}{B, A \vdash B}}{A \supset B, A \vdash B}}{\forall x.(A \supset B), A \vdash B}}{\forall x.(A \supset B), A \vdash \forall x.B}}{\forall x.(A \supset B) \vdash A \supset \forall x.B}}{\vdash \forall x.(A \supset B) \supset (A \supset \forall x.B)}$$

5 Homework 3

1) Show that LK is consistent (hint: use the cut-elimination theorem).

2) Let A be a generic formula in MS. Does $A \vdash (A \supset \bot) \supset \bot$ admit a uniform proof in MS? Does $\neg\forall x.p(x) \vdash \exists x.\neg p(x)$ admit a uniform proof in MS? Show the uniform proofs, if any, or explain why they do not exist.

3) Consider goals in the set G generated by the grammar:

$$\begin{aligned} \mathsf{G} &:= \mathsf{A} \mid \mathsf{D} \supset \mathsf{A} \mid \mathsf{G} \vee \mathsf{G}, \\ \mathsf{D} &:= \mathsf{A} \mid \mathsf{G} \supset \mathsf{A} \mid \forall x.\mathsf{D}, \end{aligned}$$

where A stands for any atom and D is the set of clauses. Define in MS a backchain rule for the language considered. Remember that a backchain rule has two properties: 1) read from bottom to top, it transforms one logic programming problem into several ones (or zero); 2) it produces only uniform proofs.

6 Solutions to Homework 3

1) We shall reason by contradiction. Suppose that LK is inconsistent: we can then find two proofs Π_1 and Π_2 for both $\vdash A$ and $\vdash \neg A$. We can then also build the following proof for $\vdash$:

$$\bowtie\,\dfrac{\dfrac{\Pi_1}{\vdash A} \qquad \bowtie\,\dfrac{\dfrac{\Pi_2}{\vdash \neg A} \qquad \neg_{\mathsf{L}}\,\dfrac{A \vdash A}{\neg A, A \vdash}}{A \vdash}}{\vdash}\;.$$

By the cut-elimination theorem a proof for $\vdash$ must exist in which the cut rule is not used. But this raises a contradiction, since $\vdash$ is not an axiom and is not the conclusion of any rule of LK different from $\bowtie$.

2a) Depending on A, the formula $A \vdash (A \supset \bot) \supset \bot$ admits or not a uniform proof. For example, if $A = a$ (*i.e.* A is an atom) one may consider the uniform proof

$$\supset_{\mathsf{R}}\,\dfrac{\supset_{\mathsf{L}}\,\dfrac{a \vdash a \qquad a, \bot \vdash \bot}{a, a \supset \bot \vdash \bot}}{a \vdash (a \supset \bot) \supset \bot}\;.$$

If one considers $A = \forall x.(p(x) \vee q(x))$ instead, there is no way of building a uniform proof. The following derivation does not lead to a uniform proof, because the premise is not provable:

$$\supset_{\mathsf{R}}\,\dfrac{\supset_{\mathsf{L}}\,\dfrac{\forall_{\mathsf{R}}\,\dfrac{\vee_{\mathsf{RL}}\,\dfrac{\forall x.(p(x) \vee q(x)) \vdash p(x)}{\forall x.(p(x) \vee q(x)) \vdash p(x) \vee q(x)}}{\forall x.(p(x) \vee q(x)) \vdash \forall x.(p(x) \vee q(x))} \qquad \forall x.(p(x) \vee q(x)), \bot \vdash \bot}{\forall x.(p(x) \vee q(x)), \forall x.(p(x) \vee q(x)) \supset \bot \vdash \bot}}{\forall x.(p(x) \vee q(x)) \vdash (\forall x.(p(x) \vee q(x)) \supset \bot) \supset \bot}\;;$$

and similarly for:

$$\supset_{\mathsf{R}} \dfrac{\supset_{\mathsf{L}} \dfrac{\forall_{\mathsf{R}} \dfrac{\vee_{\mathsf{RR}} \dfrac{\forall x.(p(x)\vee q(x)) \vdash q(x)}{\forall x.(p(x)\vee q(x)) \vdash p(x)\vee q(x)}}{\forall x.(p(x)\vee q(x)) \vdash \forall x.(p(x)\vee q(x))} \quad \forall x.(p(x)\vee q(x)),\bot \vdash \bot}{\forall x.(p(x)\vee q(x)),\forall x.(p(x)\vee q(x)) \supset \bot \vdash \bot}}{\forall x.(p(x)\vee q(x)) \vdash (\forall x.(p(x)\vee q(x)) \supset \bot) \supset \bot} .$$

An alternative is to consider the following derivation, but again we have a premise which is not provable, no matter which term t we choose (it cannot be $t = x$, of course):

$$\supset_{\mathsf{R}} \dfrac{\forall_{\mathsf{L}} \dfrac{\supset_{\mathsf{L}} \dfrac{\forall_{\mathsf{R}} \dfrac{\vee_{\mathsf{RL}} \dfrac{p(t)\vee q(t) \vdash p(x)}{p(t)\vee q(t) \vdash p(x)\vee q(x)}}{p(t)\vee q(t) \vdash \forall x.(p(x)\vee q(x))} \quad p(t)\vee q(t),\bot \vdash \bot}{p(t)\vee q(t),\forall x.(p(x)\vee q(x)) \supset \bot \vdash \bot}}{\forall x.(p(x)\vee q(x)),\forall x.(p(x)\vee q(x)) \supset \bot \vdash \bot}}{\forall x.(p(x)\vee q(x)) \vdash (\forall x.(p(x)\vee q(x)) \supset \bot) \supset \bot} ;$$

there is an analogous derivation with $\vee_{\mathsf{RR}}$ instead of $\vee_{\mathsf{RL}}$ with the same problem. Another alternative is considering this derivation:

$$\supset_{\mathsf{R}} \dfrac{\forall_{\mathsf{L}} \dfrac{\vee_{\mathsf{L}} \dfrac{\supset_{\mathsf{L}} \dfrac{p(t) \vdash \forall x.(p(x)\vee q(x)) \quad p(t),\bot \vdash \bot}{p(t),\forall x.(p(x)\vee q(x)) \supset \bot \vdash \bot} \quad \supset_{\mathsf{L}} \dfrac{q(t) \vdash \forall x.(p(x)\vee q(x)) \quad q(t),\bot \vdash \bot}{q(t),\forall x.(p(x)\vee q(x)) \supset \bot \vdash \bot}}{p(t)\vee q(t),\forall x.(p(x)\vee q(x)) \supset \bot \vdash \bot}}{\forall x.(p(x)\vee q(x)),\forall x.(p(x)\vee q(x)) \supset \bot \vdash \bot}}{\forall x.(p(x)\vee q(x)) \vdash (\forall x.(p(x)\vee q(x)) \supset \bot) \supset \bot} ;$$

again, we have premises which are not provable. This exhausts all possible cases. Please note that when $A = p \vee q$ a uniform proof for $A \vdash (A \supset \bot) \supset \bot$ *does* exist.

2b) There is no uniform proof for $\neg\forall x.p(x) \vdash \exists x.\neg p(x)$. In fact the following derivation is compulsory, if we require uniformity:

$$\exists_{\mathsf{R}} \dfrac{\supset_{\mathsf{R}} \dfrac{\supset_{\mathsf{L}} \dfrac{p(t) \vdash \forall x.p(x) \quad \bot, p(t) \vdash \bot}{\neg\forall x.p(x), p(t) \vdash \bot}}{\neg\forall x.p(x) \vdash \neg p(t)}}{\neg\forall x.p(x) \vdash \exists x.\neg p(x)} ;$$

there is no proof for its left premise, being t different from any eigenvariable we may choose in a $\forall_{\mathsf{R}}$ rule's application.

3) The most general case is the following:

$$\mathsf{b}\,\dfrac{\mathsf{P}, D \vdash G\sigma}{\mathsf{P} \vdash G_1 \vee \cdots \vee G_h} = (\vee_{\mathsf{RL}} \text{ or } \vee_{\mathsf{RR}})\,\dfrac{\vdots}{\mathsf{P} \vdash G_1 \vee \cdots \vee G_h}, \quad \text{where} \quad (\vee_{\mathsf{RL}} \text{ or } \vee_{\mathsf{RR}})\,\dfrac{\supset_{\mathsf{R}} \dfrac{\forall_{\mathsf{L}} \dfrac{\vdots}{\mathsf{P}, \forall\vec{x}.(G \supset a'), D \vdash a}}{\mathsf{P} \vdash D \supset a}}{\vdots}, \quad \forall_{\mathsf{L}} \dfrac{\supset_{\mathsf{L}} \dfrac{\mathsf{P}, D \vdash G\sigma \quad \mathsf{P}, a'\sigma, D \vdash a}{\mathsf{P}, (G \supset a')\sigma, D \vdash a}}{\vdots} ,$$

where $a'\sigma = a$. Special cases are

$$\mathsf{b}\,\frac{\mathsf{P} \vdash G\sigma}{\mathsf{P} \vdash G_1 \vee \cdots \vee G_h} = \begin{array}{r}\supset_{\mathsf{L}}\dfrac{\mathsf{P} \vdash G\sigma \quad \mathsf{P}, a'\sigma \vdash a}{\forall_{\mathsf{L}}\dfrac{\mathsf{P}, (G \supset a')\sigma \vdash a}{\vdots}} \\ \forall_{\mathsf{L}}\,\dfrac{}{\mathsf{P}, \forall\vec{x}.(G \supset a') \vdash a} \\ (\vee_{\mathsf{RL}} \text{ or } \vee_{\mathsf{RR}})\,\dfrac{}{\vdots} \\ (\vee_{\mathsf{RL}} \text{ or } \vee_{\mathsf{RR}})\,\dfrac{}{\mathsf{P} \vdash G_1 \vee \cdots \vee G_h}\end{array}$$

and

$$\mathsf{b}\,\frac{}{\mathsf{P} \vdash G_1 \vee \cdots \vee G_h} = \begin{array}{r}\forall_{\mathsf{L}}\,\dfrac{\mathsf{P}, a'\sigma, D \vdash a}{\vdots} \\ \forall_{\mathsf{L}}\,\dfrac{}{\mathsf{P}, \forall\vec{x}.a', D \vdash a} \\ \supset_{\mathsf{R}}\,\dfrac{}{\mathsf{P} \vdash D \supset a} \\ (\vee_{\mathsf{RL}} \text{ or } \vee_{\mathsf{RR}})\,\dfrac{}{\vdots} \\ (\vee_{\mathsf{RL}} \text{ or } \vee_{\mathsf{RR}})\,\dfrac{}{\mathsf{P} \vdash G_1 \vee \cdots \vee G_h}\end{array} \quad ,$$

or a combination of the last two.