University of Bath

DEPARTMENT OF COMPUTER SCIENCE EXAMINATION

CM30071: LOGIC AND ITS APPLICATIONS

Full marks will be given for correct answers to THREE questions. If you opt to answer more than the specified number of questions, you should clearly identify which of your answers you wish to have marked. In cases where you have failed to identify the correct number of answers the marker is only obliged to consider the answers in the order they appear up to the number of answers required.

1. There are three pigeons, 1, 2 and 3, and two holes, a and b, and we want to prove that if we put each pigeon into a hole, there is a hole with at least two pigeons. So, if p_{ih} stands for 'pigeon i is in hole h', we want to prove, in natural deduction, the formula

$$F = ((p_{1a} \vee p_{1b}) \wedge (p_{2a} \vee p_{2b}) \wedge (p_{3a} \vee p_{3b})) \rightarrow (\quad (p_{1a} \wedge p_{2a}) \vee (p_{1a} \wedge p_{3a}) \vee (p_{2a} \wedge p_{3a}) \vee \\ (p_{1b} \wedge p_{2b}) \vee (p_{1b} \wedge p_{3b}) \vee (p_{2b} \wedge p_{3b}) \quad) \quad .$$

(a) Show a system of natural deduction for propositional classical logic; use any notation you like, provided the system is complete. [6]

(b) Show a proof in natural deduction of $\neg G \wedge \neg H$ from hypothesis $\neg(G \vee H)$. [5]

(c) Show a proof in natural deduction of $\neg G \vee \neg H$ from hypothesis $\neg(G \wedge H)$. [6]

(d) Show a proof in natural deduction of $G \vee H$ from hypotheses $G \vee \neg K$ and $K \vee H$. [7]

(e) Show a proof in natural deduction of the formula F defined above. Hints: 1) reason by contradiction; 2) use the proofs you found in (b), (c) and (d); 3) derive both $p_{1a} \vee p_{1a}$ and $\neg p_{1a} \vee \neg p_{1a}$. [10]

2. (a) Show a sequent system for predicate classical logic; use any notation you like, provided the system is complete; remember to state the conditions on rules for quantifiers. [9]

(b) There are three pigeons, 1, 2 and 3, and two holes, a and b, and we want to prove that if we put each pigeon into a hole, there is a hole with at least two pigeons. So, if p_{ih} stands for 'pigeon i is in hole h', we have the formula

$$F = ((p_{1a} \vee p_{1b}) \wedge (p_{2a} \vee p_{2b}) \wedge (p_{3a} \vee p_{3b})) \rightarrow (\quad (p_{1a} \wedge p_{2a}) \vee (p_{1a} \wedge p_{3a}) \vee (p_{2a} \wedge p_{3a}) \vee \\ (p_{1b} \wedge p_{2b}) \vee (p_{1b} \wedge p_{3b}) \vee (p_{2b} \wedge p_{3b}) \quad) \quad .$$

Prove F in the sequent system you defined for (a). Do not give a complete, exhaustive proof (it would be huge): rather, say why the proof works, by analysing its structure. [15]

(b) Prove, in the sequent system you defined for (a), the formula

$$\forall x.\exists y.\forall z.\Big(p(x,y) \rightarrow \big(p(x,x) \rightarrow p(x,z)\big)\Big) \quad .$$

[10]

3. Consider a system $\mathcal{N}$ of natural deduction consisting of the following rules:

$$\wedge_{\mathrm{I}}\frac{F \quad G}{F \wedge G} \quad , \qquad \wedge_{\mathrm{EL}}\frac{F \wedge G}{F} \quad , \qquad \wedge_{\mathrm{ER}}\frac{F \wedge G}{G} \quad , \qquad \to_{\mathrm{I}}^{\alpha}\frac{\begin{array}{c}[F]^{\alpha}\\ \vdots \\ G\end{array}}{F \to G} \quad , \qquad \to_{\mathrm{E}}\frac{F \quad F \to G}{G} \quad .$$

We want to transform proofs in this system into proofs of a sequent system.

(a) Provide a sequent system $\mathcal{S}$ with cut for the intuitionistic logic defined over the connectives $\wedge$ and $\to$ (this means: a sequent system with rules for $\wedge$ and $\to$ such that sequents have one and only one formula at the right of the entailment symbol). [8]

(b) Describe how proofs, in $\mathcal{N}$, of formulae only containing the $\wedge$ connective can be transformed into proofs in $\mathcal{S}$, by reasoning inductively on their structure. [12]

(c) Complete the description of how proofs, in $\mathcal{N}$, (of any formula containing $\wedge$ and $\to$) can be transformed into proofs in $\mathcal{S}$, by reasoning inductively on their structure. [14]

4. (a) Define the following notions of basic modal logic:

- *Kripke model*, [3]
- the *forcing* relation, in particular for $\diamond$ and $\Box$, [4]
- the *validity* of a formula, [3]
- $A \models B$, where A and B are formulae, [2]
- *semantic equivalence* $\equiv$. [2]

(b) Prove that axiom K, *i.e.*, the formula $(\Box(F \to G) \wedge \Box F) \to \Box G$, is valid. [8]

(c) Which of the following statements are true in basic modal logic? Prove them, or disprove them with a counterexample.

$$\begin{array}{rcl} \Box(F \wedge G) & \equiv & \Box F \wedge \Box G \quad , \\ \Box(F \vee G) & \equiv & \Box F \vee \Box G \quad , \\ \diamond(F \wedge G) & \equiv & \diamond F \wedge \diamond G \quad , \\ \diamond(F \vee G) & \equiv & \diamond F \vee \diamond G \quad . \end{array}$$

[12]