University of Bath

DEPARTMENT OF COMPUTER SCIENCE EXAMINATION

CM30071: LOGIC AND ITS APPLICATIONS

Full marks will be given for correct answers to THREE questions. If you opt to answer more than the specified number of questions, you should clearly identify which of your answers you wish to have marked. In cases where you have failed to identify the correct number of answers the marker is only obliged to consider the answers in the order they appear up to the number of answers required.

1. (a) Show a system of natural deduction for propositional classical logic; use any notation you like, provided the system is complete. [4]

   (b) Let subscripted p and q be propositional variables; consider, for $h > 0$, the following formulae:

$$\begin{aligned} A_h &= p_{h-1} \to (p_h \vee q_h) \quad ; \\ B_h &= q_{h-1} \to (p_h \vee q_h) \quad ; \\ F_h &= \Big((p_0 \vee q_0) \wedge \big((A_1 \wedge B_1) \wedge \cdots \wedge (A_h \wedge B_h)\big)\Big) \to (p_h \vee q_h) \quad . \end{aligned}$$

   For example,

$$\begin{aligned} F_2 = \Big(\quad & (p_0 \vee q_0) \wedge \\ & \big((p_0 \to (p_1 \vee q_1)) \wedge (q_0 \to (p_1 \vee q_1))\big) \wedge \\ & \big((p_1 \to (p_2 \vee q_2)) \wedge (q_1 \to (p_2 \vee q_2))\big)\Big) \\ \to & (p_2 \vee q_2) \quad . \end{aligned}$$

   Prove that every F_h is a tautology by using the system of natural deduction that you provided before. Note: to get full marks, it is necessary to provide a concise solution; please feel free to define pieces of derivations you might need, give them a name and then use the names instead of replicating many derivations. [16]

2. (a) Show a sequent system for predicate classical logic; use any notation you like, provided the system is complete. [7]

   (b) Prove, in the system you gave, without using any cut rule, that

$$\exists x.\forall y.\exists z.\Big(p(x) \to \big(q(z) \to p(y)\big)\Big) \quad .$$

   [13]

3. Let **HT** (for Hilbert-Tarski) be the deductive system whose axiom schemes are:

$$\begin{aligned} H_1 &= A \to (B \to A) \quad , \\ H_2 &= (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \quad , \\ H_3 &= (\neg B \to \neg A) \to ((\neg B \to A) \to B) \quad , \end{aligned}$$

and whose inference rule is *modus ponens*:

$$\mathsf{mp}\,\frac{A \quad A \to B}{B} \quad .$$

We know that this system is complete for propositional classical logic. The purpose of this exercise is to prove that Gentzen system **LK** is complete, by reducing proofs in **HT** to proofs in **LK**.

(a) Show a fragment of **LK** which only deals with the $\to$ and $\neg$ connectives (note: fragment means a subset of the rules of the entire system as we have studied it). This fragment will have to include the cut rule. [4]

(b) Show how every axiom of **HT** can be proved in the fragment of **LK** you showed before. [6]

(c) Make an inductive argument on a given proof of **HT** and show how to transform it into a proof in the fragment of **LK**. Please note that you will need to use the cut rule to deal with mp. [10]

4. (a) Define the following notions of basic modal logic:
   - *Kripke model*,
   - the *forcing* relation $\models$, in particular for $\Diamond$ and $\Box$,
   - $A \models B$, where A and B are formulae,
   - *semantic equivalence* $\equiv$.

   [8]

   (b) Which of the following statements are true in basic modal logic? Prove them, or disprove them with a counterexample.

$$\begin{aligned} \Diamond\neg A &\equiv \neg\Box A \quad , \\ \Box\neg A &\equiv \neg\Diamond A \quad , \\ \Box(A \wedge B) &\equiv \Box A \wedge \Box B \quad , \\ \Box(A \vee B) &\equiv \Box A \vee \Box B \quad , \\ \Diamond(A \wedge B) &\equiv \Diamond A \wedge \Diamond B \quad , \\ \Diamond(A \vee B) &\equiv \Diamond A \vee \Diamond B \quad . \end{aligned}$$

[12]