CM20019—COMPUTATION III:
FORMAL LOGIC AND SEMANTICS
COURSEWORK, DUE 22.11.2007, WITH SOLUTIONS

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Information and instructions for the coursework submission process:

- This coursework is to be submitted through the coursework post box located outside 1W2.23 by 22 November 2007 at 17:00. Please be careful to select the box relative to this course.
- This coursework is to be submitted in written form, on paper, bound securely. Please make sure it contains information to identify you, clearly written: surname, first name, and student number. You must include the coursework signature sheet.
- This coursework will contribute 25% of the final grade for this unit.
- This is an individual coursework, and it should be completed within the student's own time.
- Feedback will be provided by example solutions.

Plagiarism will be detected and will be penalised. Remember that, while there are not many ways to answer a problem correctly, there are infinitely many, **unique** ways of making mistakes. Remember also that **even the best make mistakes**.

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Problem 1. Use the unification algorithm on the following pairs of terms and either find most general unifiers or tell when the terms are not unifiable and why.

(a) $f(X,Y)$ and $f(Z,X)$; [2 points]
(b) $f(X,Y)$ and $g(Z,X)$; [2 points]
(c) $f(X,X)$ and $f(Y,g(Y))$; [3 points]
(d) $f(X,Y,g(g(Z)))$ and $f(g(Y),g(Z),X)$; [4 points]
(e) $f(X,g(a))$ and $f(Y,X)$. [4 points]

Solution.

(a) $[X := Z, Y := Z]$.
(b) Not unifiable—clash of function symbols.
(c) Not unifiable—occur check.
(d) $[X := g(g(Z)), Y := g(Z)]$.
(e) $[X := g(a), Y := g(a)]$.

Problem 2. Either prove the validity, by resorting to relevant definitions, or disprove it, by providing a counter-model, of the following formulae:

(a) $F = (\forall X)(\exists Y)(p(X) \to p(Y))$; [6 points]
(b) $G = (\exists X)(\forall Y)(p(X) \to p(Y))$. [9 points]

Then, answer the question:

(c) Are F and G equivalent? [5 points]

(*Note*: a counter-model for a formula is an interpretation in which the formula is false; two formulae are equivalent if whenever one is true in an interpretation, the other is true in that interpretation as well.)

Solution.

(a) Valid. For every interpretation, choose a valuation where Y is interpreted as X, so that premiss and conclusion of the implication are both true or both false.

(b) Valid. Take any interpretation: if $p(Y)$ is true, for any valuation of Y the conclusion of the implication is true, and so the formula is. Otherwise, choose an X such that $p(X)$ is false, and so the premiss of the implication is false, and the formula is true.

(c) Yes, because F and G are both valid.

Problem 3. We are given the predicate symbols p and q, and the function symbol f, all of which have arity 1. Let I be an interpretation such that:

- its domain is the set $\mathbb{N}$ of natural numbers (including 0);
- p is interpreted as $\{\, 3n \mid n \in \mathbb{N} \,\} \subset \mathbb{N}$;
- q is interpreted as $\{\, n \mid n \text{ is prime} \,\} \subset \mathbb{N}$;
- f is interpreted as the function $\mathbf{f}\colon \mathbb{N} \to \mathbb{N}$ such that $\mathbf{f}(n) = n + 6$.

Please answer the following questions in concise, not necessarily formal, but precise and unambiguous mathematical terms.

(a) Is the formula $p(f(X)) \wedge q(X)$ false in I for some valuation? If so, which one? If not, why not? [2 points]

(b) Is the formula $p(f(X)) \wedge q(X)$ true in I for some valuation? If so, which one? If not, why not? [2 points]

(c) Is the formula $\neg(\exists X)(p(f(X)) \wedge q(X))$ true in I? Explain. [2 points]

(d) Is the formula $(\forall X)(q(X) \to (p(f(X)) \vee q(f(X))))$ true in I? Explain. [5 points]

(e) Is the formula $(\forall X)(q(X) \to (p(f(X)) \vee q(f(X))))$ valid? Explain. [4 points]

(f) Find, in the given language, a satisfiable formula F such that it contains the symbol f and such that I is a model for

$$(\forall X)(F \to (p(X) \wedge q(X))) \quad .$$

[11 points]

(g) Find, in the given language, a formula F such that it contains the symbol f and such that

$$(\forall X)(F \to (p(X) \wedge q(X)))$$

is valid. [9 points]

(*Note*: a formula F is satisfiable if there is an interpretation and a valuation for which F is true; the set of prime numbers is $\{2, 3, 5, 7, 11, 13, 17, \dots\}$.)

Solution.

(a) Yes, for example in $[X := 0]$.

(b) Yes, for example in $[X := 3]$.

(c) No, because of (b).

(d) No, for example for $[X := 2]$.

(e) No, for example because of (d).

(f) For example, $F = p(f(X)) \wedge q(X)$.

(g) For example, $F = p(f(X)) \wedge \neg p(f(X))$ (note that this formula is not satisfiable).

Problem 4. Consider the following Prolog program:

```
a(0,X,X).
a(s(X),Y,s(Z)) :- a(X,Y,Z).

b(0,0).
b(s(0),s(0)).
b(s(s(0)),s(0)).
b(s(s(s(Y))),X) :- b(Y,Z),
                   b(s(Y),W),
                   b(s(s(Y)),U),
                   a(Z,W,V),
                   a(V,U,X).
c(X) :- b(Y,X).

d(X) :- b(X,X).
```

In your answers, please abbreviate any string of the kind `s(`$\cdots$`s(0)`$\cdots$`)` with the numeral corresponding to the number of its `s` symbols; for example, please abbreviate `s(s(s(0)))` as 3. Please explain the meaning of predicates in concise, not necessarily formal, but precise and unambiguous mathematical terms.

(a) What is the meaning of the predicate `a(X,Y,Z)`? [6 points]
(b) What is the meaning of the predicate `b(X,Y)`? [16 points]
(c) What is the meaning of the predicate `c(X)`? [4 points]
(d) What is the meaning of the predicate `d(X)`? [4 points]

Solution.

(a) `a(X,Y,Z)` iff $\mathtt{X} + \mathtt{Y} = \mathtt{Z}$.
(b) `b(X,Y)` iff `Y` is the `X`th number in the succession

$$S = \langle 0, 1, 1, 2, 4, 7, 13, \dots \rangle \quad ,$$

where 0 is the 0th number and each number is the sum of the three ones that immediately precede it.
(c) `c(X)` iff $\mathtt{X} \in S$.
(d) `d(X)` iff `X` is the `X`th number in S. There are three possibilities: `d(0)`, `d(1)` and `d(4)`.